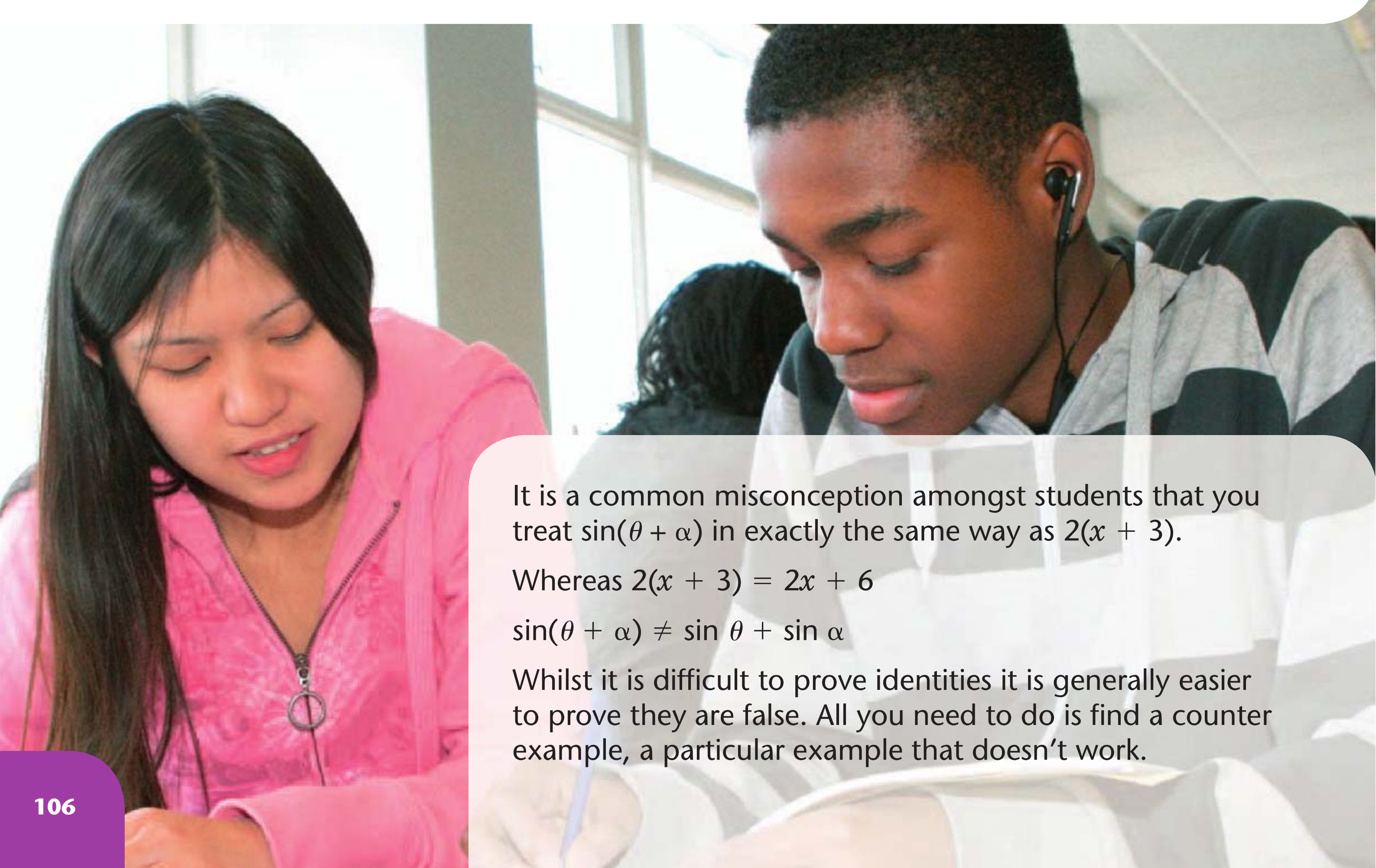
7

After completing this chapter you should be able to

1. use the addition formulae
2. use the double angle formulae
3. write expressions of the form $a\cos\theta \pm b\sin\theta$ in the form $R\cos(\theta \pm \alpha)$ and/or $R\sin(\theta \pm \alpha)$
4. use the factor formulae
5. use all of the above to solve equations and prove identities.

Further trigonometric identities and their applications



It is a common misconception amongst students that you treat $\sin(\theta + \alpha)$ in exactly the same way as $2(x + 3)$.

Whereas $2(x + 3) = 2x + 6$

$\sin(\theta + \alpha) \neq \sin\theta + \sin\alpha$

Whilst it is difficult to prove identities it is generally easier to prove they are false. All you need to do is find a counter example, a particular example that doesn't work.

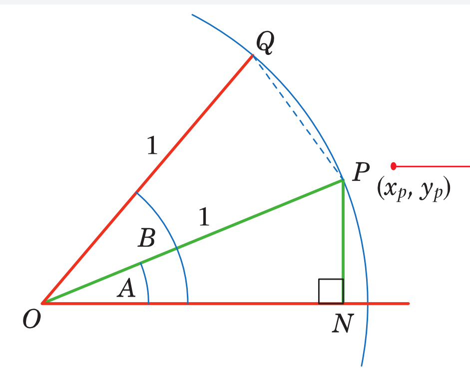
7.1 You need to know and be able to use the addition formulae.

- $\sin(A + B) \equiv \sin A \cos B + \cos A \sin B$ $\quad\quad$ $\sin(A - B) \equiv \sin A \cos B - \cos A \sin B$
- $\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$ $\quad\quad$ $\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$
- $\tan(A + B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ $\quad\quad$ $\tan(A - B) \equiv \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

Although you will not be expected to derive these formulae from first principles, it will help your understanding of them to see how one of them can be derived.

Example 1

Show that $\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$



Draw a circle, centre the origin and with unit radius. Place points P and Q on the circumference such that OP and OQ make angles A and B with the x-axis, as shown.

The coordinates of P are $(\cos A, \sin A)$ and those of Q are $(\cos B, \sin B)$.

In $\triangle PON$ $\cos A = \frac{x_P}{1}$, $\sin A = \frac{y_P}{1}$, as the radius has length 1 (unit radius).

So

$$PQ^2 = (\cos B - \cos A)^2 + (\sin B - \sin A)^2$$

Use the formula for the distance between two points: $PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

$$\begin{aligned} &= (\cos^2 B - 2\cos A \cos B + \cos^2 A) \\ &\quad + (\sin^2 B - 2\sin A \sin B + \sin^2 A) \\ &= (\sin^2 A + \cos^2 A) + (\sin^2 B + \cos^2 B) \\ &\quad - 2(\cos A \cos B + \sin A \sin B) \\ &= 2 - 2(\cos A \cos B + \sin A \sin B) \end{aligned}$$

Use $\sin^2 A + \cos^2 A \equiv 1$ and $\sin^2 B + \cos^2 B \equiv 1$.

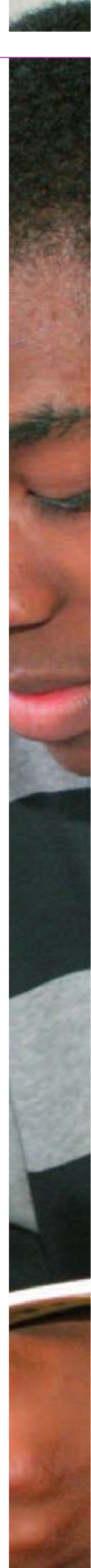
But $\quad PQ^2 = 1^2 + 1^2 - 2\cos(A - B)$
$\quad\quad\quad = 2 - 2\cos(A - B)$

Using the cosine rule in $\triangle POQ$, with $OP = OQ = 1$ and $\angle POQ = (A - B)$.

Comparing the two results for PQ^2

$\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$

The other formulae involving sine and cosine can be constructed using the one in the example above.



Example 2

Use the result in Example 1 to show that:

a $\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$

b $\sin(A + B) \equiv \sin A \cos B + \cos A \sin B$

c $\sin(A - B) \equiv \sin A \cos B - \cos A \sin B$

a Replace B by $(-B)$ in

$$\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$$

So $\cos(A + B) \equiv \cos A \cos(-B) + \sin A \sin(-B)$

$\therefore \quad \cos(A + B) \equiv \cos A \cos B - \sin A \sin B$

Use the results $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$ (See Chapter 8 in Book C2.)

b Replace A by $\left(\frac{\pi}{2} - A\right)$ in

$$\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$$

So $\cos\left[\left(\frac{\pi}{2} - A\right) - B\right] \equiv \cos\left(\frac{\pi}{2} - A\right)\cos B + \sin\left(\frac{\pi}{2} - A\right)\sin B$

$\cos\left[\frac{\pi}{2} - (A + B)\right] \equiv \cos\left(\frac{\pi}{2} - A\right)\cos B + \sin\left(\frac{\pi}{2} - A\right)\sin B$

$\therefore \quad \sin(A + B) \equiv \sin A \cos B + \cos A \sin B$

Use $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ and $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$ (See Chapter 8 in Book C2.)

c Replace B by $(-B)$ in the result in **b**:

$\sin(A - B) \equiv \sin A \cos B + \cos A \sin(-B)$

so $\sin(A - B) \equiv \sin A \cos B - \cos A \sin B$

To find similar expressions for $\tan(A + B)$ and $\tan(A - B)$ you can use the fact that $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and divide the appropriate results given above.

Example 3

Show that

a $\tan(A + B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

b $\tan(A - B) \equiv \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

a $\tan(A+B) \equiv \dfrac{\sin(A+B)}{\cos(A+B)}$

$\equiv \dfrac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$

Dividing the 'top and bottom' by $\cos A\cos B$ gives

$\tan(A+B)$

$$\equiv \dfrac{\dfrac{\sin A\cancel{\cos B}}{\cos A\cancel{\cos B}} + \dfrac{\cancel{\cos A}\sin B}{\cancel{\cos A}\cos B}}{\dfrac{\cancel{\cos A}\cancel{\cos B}}{\cancel{\cos A}\cancel{\cos B}} - \dfrac{\sin A\sin B}{\cos A\cos B}}$$

Cancel terms, as shown, and use the result $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$$\equiv \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$$

b Replace B by $-B$ in the result above:

$$\tan(A-B) \equiv \dfrac{\tan A + \tan(-B)}{1 - \tan A\tan(-B)}$$

Use the result $\tan(-\theta) = -\tan\theta$. See Chapter 8 in Book C2.

$$\equiv \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$$

Example 4

Show, using the formula for $\sin(A-B)$, that $\sin 15° = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

$\sin 15° = \sin(45 - 30)°$

You know the exact form of sin and cos for many angles, e.g. 30°, 45°, 60°, 90°, 180°…, so write 15° using two of these angles. [You could equally use sin(60 − 45)°.]

$= \sin 45°\cos 30° - \cos 45°\sin 30°$

$= (\tfrac{1}{2}\sqrt{2})(\tfrac{1}{2}\sqrt{3}) - (\tfrac{1}{2}\sqrt{2})(\tfrac{1}{2})$

$= \tfrac{1}{4}(\sqrt{3}\sqrt{2} - \sqrt{2})$

$= \tfrac{1}{4}(\sqrt{6} - \sqrt{2})$

Example 5

Given that $\sin A = -\tfrac{3}{5}$ and $180° < A < 270°$, and that $\cos B = -\tfrac{12}{13}$ and B is obtuse, find the value of

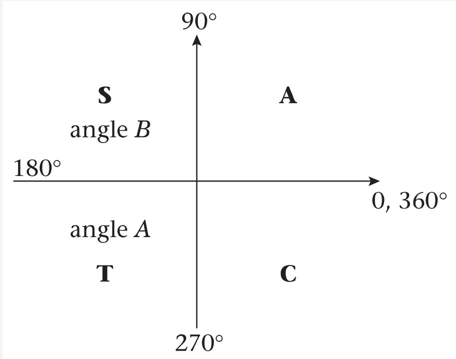
a $\cos(A-B)$

b $\tan(A+B)$

a $\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$

$\cos^2 A \equiv 1 - \sin^2 A = 1 - \left(-\frac{3}{5}\right)^2 = \frac{16}{25}$

So $\cos A = \pm\frac{4}{5}$



but A is in the third quadrant, where $\cos$ is $-$ve,

$\therefore \quad \cos A = -\frac{4}{5}$

$\sin^2 B \equiv 1 - \cos^2 B = 1 - \left(-\frac{12}{13}\right)^2 = \frac{25}{169}$

So $\sin B = \pm\frac{5}{13}$

but B is in the 2nd quadrant

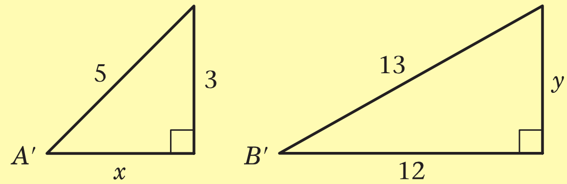
$\therefore \quad \sin B = +\frac{5}{13}$

$\cos(A - B) = \left(-\frac{4}{5}\right)\left(-\frac{12}{13}\right) + \left(-\frac{3}{5}\right)\left(+\frac{5}{13}\right)$

$= \frac{33}{65}$

You need to find $\cos A$ and $\sin B$.
Take note of the quadrants that A and B are in.

You can also work with the associated acute angles A' and B'. Draw right-angled triangles and use Pythagoras' theorem.



$x^2 = 5^2 - 3^2$	$y^2 = 13^2 - 12^2$
so $x = 4$	so $y = 5$
As A is in 3rd quadrant.	As B is in 2nd quadrant.
$\sin A = -\sin A' = -\frac{3}{5}$	$\sin B = +\sin B' = +\frac{5}{13}$
$\cos A = -\cos A' = -\frac{4}{5}$	$\cos B = -\cos B' = -\frac{12}{13}$
$\tan A = +\tan A' = +\frac{3}{4}$	$\tan B = -\tan B' = -\frac{5}{12}$

b $\tan(A + B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

So $\tan(A + B) = \dfrac{\frac{3}{4} + \left(-\frac{5}{12}\right)}{1 - \left(\frac{3}{4}\right)\left(-\frac{5}{12}\right)}$

$= \dfrac{\frac{1}{3}}{\frac{63}{48}} = \frac{1}{3} \times \frac{48}{63} = \frac{16}{63}$

You can use the above results for $\tan A$ and $\tan B$, or use $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ with the results for $\sin A$, $\cos A$, $\sin B$ and $\cos B$.

Example 6

Given that $2\sin(x + y) = 3\cos(x - y)$, express $\tan x$ in terms of $\tan y$.

Expanding $\sin(x + y)$ and $\cos(x - y)$ gives

$2\sin x \cos y + 2\cos x \sin y$

$= 3\cos x \cos y + 3\sin x \sin y$

so $\dfrac{2\sin x \cos y}{\cos x \cos y} + \dfrac{2\cos x \sin y}{\cos x \cos y}$

$= \dfrac{3\cos x \cos y}{\cos x \cos y} + \dfrac{3\sin x \sin y}{\cos x \cos y}$

This is similar to the expression seen in deriving $\tan(A + B)$. A good strategy is to divide both sides by $\cos x \cos y$.

$2\tan x + 2\tan y = 3 + 3\tan x \tan y$

$2\tan x - 3\tan x \tan y = 3 - 2\tan y$

Collect all terms in $\tan x$ on one side.

$\tan x(2 - 3\tan y) = 3 - 2\tan y$

Factorise.

So $\quad \tan x = \dfrac{3 - 2\tan y}{2 - 3\tan y}$

Exercise 7A

1 A student makes the mistake of thinking that $\sin(A + B) \equiv \sin A + \sin B$.
Choose non-zero values of A and B to show that this statement is not true for all values of A and B.

> This is a very common error – don't make the same mistake. One counterexample is sufficient to disprove a statement.

2 Using the expansion of $\cos(A - B)$ with $A = B = \theta$, show that $\sin^2\theta + \cos^2\theta \equiv 1$.

3 **a** Use the expansion of $\sin(A - B)$ to show that $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$.

b Use the expansion of $\cos(A - B)$ to show that $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$.

4 Express the following as a single sine, cosine or tangent:

a $\sin 15° \cos 20° + \cos 15° \sin 20°$

b $\sin 58° \cos 23° - \cos 58° \sin 23°$

c $\cos 130° \cos 80° - \sin 130° \sin 80°$

d $\dfrac{\tan 76° - \tan 45°}{1 + \tan 76° \tan 45°}$

e $\cos 2\theta \cos\theta + \sin 2\theta \sin\theta$

f $\cos 4\theta \cos 3\theta - \sin 4\theta \sin 3\theta$

g $\sin \tfrac{1}{2}\theta \cos 2\tfrac{1}{2}\theta + \cos \tfrac{1}{2}\theta \sin 2\tfrac{1}{2}\theta$

h $\dfrac{\tan 2\theta + \tan 3\theta}{1 - \tan 2\theta \tan 3\theta}$

i $\sin(A + B)\cos B - \cos(A + B)\sin B$

j $\cos\left(\dfrac{3x + 2y}{2}\right)\cos\left(\dfrac{3x - 2y}{2}\right) - \sin\left(\dfrac{3x + 2y}{2}\right)\sin\left(\dfrac{3x - 2y}{2}\right)$

5 Calculate, without using your calculator, the exact value of:

a $\sin 30° \cos 60° + \cos 30° \sin 60°$

b $\cos 110° \cos 20° + \sin 110° \sin 20°$

c $\sin 33° \cos 27° + \cos 33° \sin 27°$

d $\cos\frac{\pi}{8}\cos\frac{\pi}{8} - \sin\frac{\pi}{8}\sin\frac{\pi}{8}$

e $\sin 60° \cos 15° - \cos 60° \sin 15°$

f $\cos 70°(\cos 50° - \tan 70° \sin 50°)$

g $\dfrac{\tan 45° + \tan 15°}{1 - \tan 45° \tan 15°}$

h $\dfrac{1 - \tan 15°}{1 + \tan 15°}$

> **Hint:** $\tan 45° = 1$.

i $\dfrac{\tan\left(\frac{7\pi}{12}\right) - \tan\left(\frac{\pi}{3}\right)}{1 + \tan\left(\frac{7\pi}{12}\right)\tan\left(\frac{\pi}{3}\right)}$

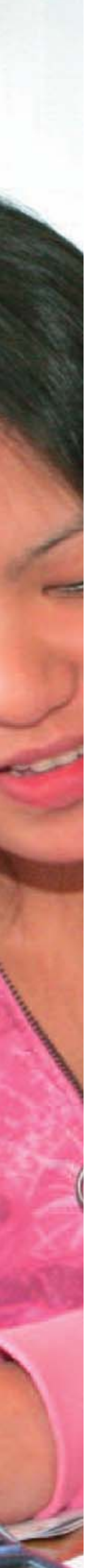
j $\sqrt{3}\cos 15° - \sin 15°$

> **Hint:** Look at **e**.

6 Triangle ABC is such that $AB = 3$ cm, $BC = 4$ cm, $\angle ABC = 120°$ and $\angle BAC = \theta°$.

a Write down, in terms of θ, an expression for $\angle ACB$.

b Using the sine rule, or otherwise, show that $\tan\theta° = \dfrac{2\sqrt{3}}{5}$.



7 Prove the identities

a $\sin(A + 60°) + \sin(A - 60°) \equiv \sin A$

b $\dfrac{\cos A}{\sin B} - \dfrac{\sin A}{\cos B} \equiv \dfrac{\cos(A + B)}{\sin B \cos B}$

c $\dfrac{\sin(x + y)}{\cos x \cos y} \equiv \tan x + \tan y$

d $\dfrac{\cos(x + y)}{\sin x \sin y} + 1 \equiv \cot x \cot y$

e $\cos\left(\theta + \dfrac{\pi}{3}\right) + \sqrt{3}\sin\theta \equiv \sin\left(\theta + \dfrac{\pi}{6}\right)$

f $\cot(A + B) \equiv \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$

g $\sin^2(45 + \theta)° + \sin^2(45 - \theta)° \equiv 1$

h $\cos(A + B)\cos(A - B) \equiv \cos^2 A - \sin^2 B$

8 Given that $\sin A = \frac{4}{5}$ and $\sin B = \frac{1}{2}$, where A and B are both acute angles, calculate the exact values of

a $\sin(A + B)$ **b** $\cos(A - B)$ **c** $\sec(A - B)$

9 Given that $\cos A = -\frac{4}{5}$, and A is an obtuse angle measured in radians, find the exact value of

a $\sin A$ **b** $\cos(\pi + A)$ **c** $\sin\left(\dfrac{\pi}{3} + A\right)$ **d** $\tan\left(\dfrac{\pi}{4} + A\right)$

10 Given that $\sin A = \frac{8}{17}$, where A is acute, and $\cos B = -\frac{4}{5}$, where B is obtuse, calculate the exact value of

a $\sin(A - B)$ **b** $\cos(A - B)$ **c** $\cot(A - B)$

11 Given that $\tan A = \frac{7}{24}$, where A is reflex, and $\sin B = \frac{5}{13}$, where B is obtuse, calculate the exact value of

a $\sin(A + B)$ **b** $\tan(A - B)$ **c** $\text{cosec}(A + B)$

12 Write the following as a single trigonometric function, assuming that θ is measured in radians:

a $\cos^2\theta - \sin^2\theta$ **b** $2\sin 4\theta \cos 4\theta$ **c** $\dfrac{1 + \tan\theta}{1 - \tan\theta}$ **d** $\dfrac{1}{\sqrt{2}}(\sin\theta + \cos\theta)$

13 Solve, in the interval $0° \leqslant \theta < 360°$, the following equations. Give answers to the nearest 0.1°.

a $3\cos\theta = 2\sin(\theta + 60°)$ **b** $\sin(\theta + 30°) + 2\sin\theta = 0$

c $\cos(\theta + 25°) + \sin(\theta + 65°) = 1$ **d** $\cos\theta = \cos(\theta + 60°)$

e $\tan(\theta - 45°) = 6\tan\theta$ **f** $\sin\theta + \cos\theta = 1$

Hint for part **f:** Multiply each term by $\dfrac{1}{\sqrt{2}}$

14 **a** Solve the equation $\cos\theta\cos 30° - \sin\theta\sin 30° = 0.5$, for $0 \leqslant \theta \leqslant 360°$.

b Hence write down, in the same interval, the solutions of $\sqrt{3}\cos\theta - \sin\theta = 1$.

15 **a** Express $\tan(45 + 30)°$ in terms of $\tan 45°$ and $\tan 30°$.

b Hence show that $\tan 75° = 2 + \sqrt{3}$.

16 Show that $\sec 105° = -\sqrt{2}(1 + \sqrt{3})$

17 Calculate the exact values of

a $\cos 15°$ **b** $\sin 75°$

c $\sin(120 + 45)°$ **d** $\tan 165°$

Hint for part **a:** Write 15° as $(45 - 30)°$

18 **a** Given that $3\sin(x - y) - \sin(x + y) = 0$, show that $\tan x = 2\tan y$.

b Solve $3\sin(x - 45°) - \sin(x + 45°) = 0$, for $0 \leqslant x \leqslant 360°$.

19 Given that $\sin x(\cos y + 2\sin y) = \cos x(2\cos y - \sin y)$, find the value of $\tan(x + y)$.

20 Given that $\tan(x - y) = 3$, express $\tan y$ in terms of $\tan x$.

21 In each of the following, calculate the exact value of $\tan x°$.

a $\tan(x - 45)° = \frac{1}{4}$

b $\sin(x - 60)° = 3\cos(x + 30)°$

c $\tan(x - 60)° = 2$

22 Given that $\tan A° = \frac{1}{5}$ and $\tan B° = \frac{2}{3}$, calculate, without using your calculator, the value of $A + B$,

a where A and B are both acute,

b where A is reflex and B is acute.

23 Given that $\cos y = \sin(x + y)$, show that $\tan y = \sec x - \tan x$.

24 Given that $\cot A = \frac{1}{4}$ and $\cot(A + B) = 2$, find the value of $\cot B$.

25 Given that $\tan\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$, show that $\tan x = 8 - 5\sqrt{3}$.

7.2 You can express sin 2*A*, cos 2*A* and tan 2*A* in terms of angle *A*, using the double angle formulae.

- $\sin 2A \equiv 2\sin A\cos A$
- $\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$
- $\tan 2A \equiv \dfrac{2\tan A}{1 - \tan^2 A}$

You can derive these results from the addition formulae.

Example 7

Use the expansion of $\sin(A + B)$ to show that $\sin 2A \equiv 2\sin A\cos A$

Using $\sin(A + B) \equiv \sin A\cos B + \cos A\sin B$

$\sin 2A \equiv \sin A\cos A$

$+ \cos A\sin A$

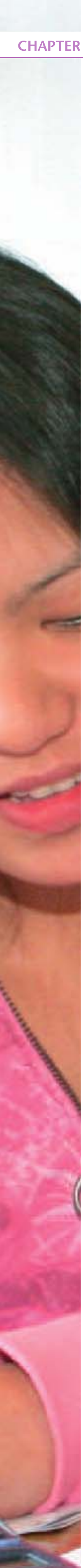
$\equiv 2\sin A\cos A$

Replace *B* by *A*.

Example 8

a By using an appropriate addition formula show that $\cos 2A \equiv \cos^2 A - \sin^2 A$.

b Hence derive the alternative forms $\cos 2A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$.



a Using $\cos(A + B)$

$\equiv \cos A \cos B - \sin A \sin B$

So $\cos 2A$

$\equiv \cos A \cos A - \sin A \sin A$

$\equiv \cos^2 A - \sin^2 A$

b $\cos 2A$

$\equiv \cos^2 A - \sin^2 A$

$\equiv \cos^2 A - (1 - \cos^2 A)$

$\equiv 2\cos^2 A - 1$

OR

$\equiv 2(1 - \sin^2 A) - 1$

$\equiv 1 - 2\sin^2 A$

Replace B with A.

Use $\sin^2 A + \cos^2 A \equiv 1$.

You can express $\cos 2A$ in terms of $\cos^2 A$ and $\sin^2 A$, or $\cos^2 A$ only, or $\sin^2 A$ only.

Example 9

Express $\tan 2A$ in terms of tan A.

Using $\tan(A + B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$\tan 2A \equiv \dfrac{2\tan A}{1 - \tan^2 A}$

Replace B by A.

Example 10

Rewrite the following expressions as a single trigonometric function:

a $2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\theta$

b $1 + \cos 4\theta$

a $2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \equiv \sin\theta$

So $2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\theta \equiv \sin\theta\cos\theta$

$\equiv \frac{1}{2}\sin 2\theta$

b $\cos 4\theta \equiv 2\cos^2 2\theta - 1$

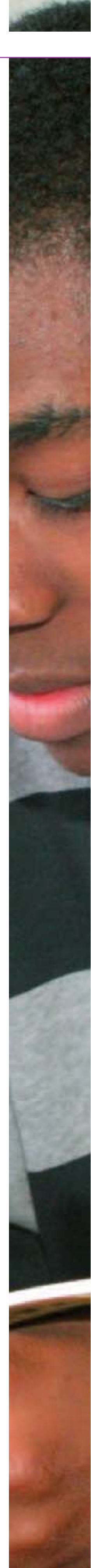
So $1 + \cos 4\theta \equiv 2\cos^2 \theta$

Using $2\sin A \cos A \equiv \sin 2A$ with $A = \frac{\theta}{2}$.

The double angle formulae allow you to convert an angle into its half angle, and vice versa. In a question it will not always be obvious that the double angle formulae are needed (i.e.you will not always see **2**A or **2**θ).

$\sin 2\theta = 2\sin\theta\cos\theta$.

Using $\cos 2A = 2\cos^2 A - 1$ with $A = 2\theta$. Choose this form of $\cos 2A$ because the -1 will cancel out the $+1$ in '$1 + \cos 4\theta$'.

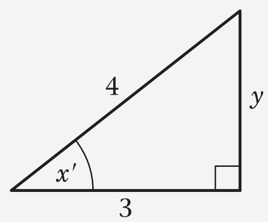


Example 11

Given that $\cos x = \frac{3}{4}$, and that $180° < x < 360°$, find the exact values of

a $\sin 2x$ **b** $\tan 2x$

a



Draw the right-angled triangle with $\cos x' = \frac{3}{4}$.

Using Pythagoras' theorem

$$y^2 = 4^2 - 3^2 = 7$$

So $y = \sqrt{7}$

$$\therefore \quad \sin x' = \frac{\sqrt{7}}{4} \text{ and } \tan x' = \frac{\sqrt{7}}{3}$$

$$\Rightarrow \quad \sin x = -\frac{\sqrt{7}}{4} \text{ and } \tan x = -\frac{\sqrt{7}}{3}$$

As $\cos x$ is +ve and x is reflex, x must be in the 4th quadrant, so $\sin x = -\sin x'$ and $\tan x = -\tan x'$.

$$\sin 2x = 2 \sin x \cos x$$

$$= 2\left(-\frac{\sqrt{7}}{4}\right)\left(\frac{3}{4}\right) = -\frac{3\sqrt{7}}{8}$$

b

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x} = \frac{-\frac{2\sqrt{7}}{3}}{1 - \frac{7}{9}}$$

$$= -\frac{2\sqrt{7}}{3} \times \frac{9}{2}$$

$$= -3\sqrt{7}$$

Exercise 7B

In equations, give answers to 1 decimal place where appropriate.

1 Write the following expressions as a single trigonometric ratio:

a $2 \sin 10° \cos 10°$

b $1 - 2\sin^2 25°$

c $\cos^2 40° - \sin^2 40°$

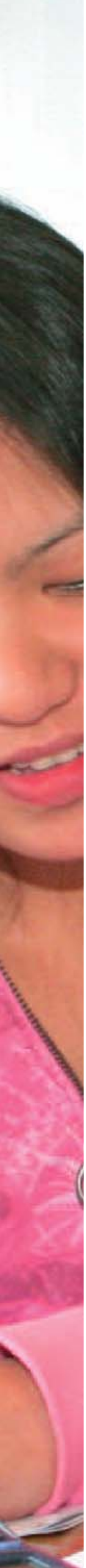
d $\dfrac{2\tan 5°}{1 - \tan^2 5°}$

e $\dfrac{1}{2\sin(24\frac{1}{2})° \cos(24\frac{1}{2})°}$

f $6\cos^2 30° - 3$

g $\dfrac{\sin 8°}{\sec 8°}$

h $\cos^2 \dfrac{\pi}{16} - \sin^2 \dfrac{\pi}{16}$



2 Without using your calculator find the exact values of:

a $2\sin(22\frac{1}{2})°\cos(22\frac{1}{2})°$ **b** $2\cos^2 15° - 1$ **c** $(\sin 75° - \cos 75°)^2$ **d** $\dfrac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$

3 Write the following in their simplest form, involving only one trigonometric function:

a $\cos^2 3\theta - \sin^2 3\theta$ **b** $6\sin 2\theta\cos 2\theta$ **c** $\dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

d $2 - 4\sin^2\dfrac{\theta}{2}$ **e** $\sqrt{1+\cos 2\theta}$ **f** $\sin^2\theta\cos^2\theta$

g $4\sin\theta\cos\theta\cos 2\theta$ **h** $\dfrac{\tan\theta}{\sec^2\theta - 2}$ **i** $\sin^4\theta - 2\sin^2\theta\cos^2\theta + \cos^4\theta$

4 Given that $\cos x = \frac{1}{4}$, find the exact value of $\cos 2x$.

5 Find the possible values of $\sin\theta$ when $\cos 2\theta = \frac{23}{25}$.

6 Given that $\cos x + \sin x = m$ and $\cos x - \sin x = n$, where m and n are constants, write down, in terms of m and n, the value of $\cos 2x$.

7 Given that $\tan\theta = \frac{3}{4}$, and that θ is acute:

a Find the exact value of **i** $\tan 2\theta$ **ii** $\sin 2\theta$ **iii** $\cos 2\theta$

b Deduce the value of $\sin 4\theta$.

8 Given that $\cos A = -\frac{1}{3}$, and that A is obtuse:

a Find the exact value of **i** $\cos 2A$ **ii** $\sin A$ **iii** $\operatorname{cosec} 2A$

b Show that $\tan 2A = \dfrac{4\sqrt{2}}{7}$.

9 Given that $\pi < \theta < \dfrac{3\pi}{2}$, find the value of $\tan\dfrac{\theta}{2}$ when $\tan\theta = \frac{3}{4}$.

10 In $\triangle ABC$, $AB = 4$ cm, $AC = 5$ cm, $\angle ABC = 2\theta$ and $\angle ACB = \theta$. Find the value of θ, giving your answer, in degrees, to 1 decimal place.

11 In $\triangle PQR$, $PQ = 3$ cm, $PR = 6$ cm, $QR = 5$ cm and $\angle QPR = 2\theta$.

a Use the cosine rule to show that $\cos 2\theta = \frac{5}{9}$.

b Hence find the exact value of $\sin\theta$.

12 The line l, with equation $y = \frac{3}{4}x$, bisects the angle between the x-axis and the line $y = mx$, $m > 0$. Given that the scales on each axis are the same, and that l makes an angle θ with the x-axis,

a write down the value of $\tan\theta$.

b Show that $m = \frac{24}{7}$.

7.3 The double angle formulae allow you to solve more equations and prove more identities.

Example 12

Prove the identity $\tan 2\theta \equiv \dfrac{2}{\cot\theta - \tan\theta}$

Start on LHS with $\tan 2\theta \equiv \dfrac{2\tan\theta}{1 - \tan^2\theta}$

Divide 'top and bottom' by $\tan\theta$.

So $\tan 2\theta \equiv \dfrac{2}{\dfrac{1}{\tan\theta} - \tan\theta}$

$\equiv \dfrac{2}{\cot\theta - \tan\theta}$

There are many starting points here; the more you know the more options you have.

Try starting with $\tan 2\theta \equiv \dfrac{\sin 2\theta}{\cos 2\theta}$ and use the double angle formulae for $\sin 2\theta$ and $\cos 2\theta$ (a bit harder!), or start with the RHS.

Example 13

By expanding $\sin(2A + A)$ show that $\sin 3A \equiv 3\sin A - 4\sin^3 A$.

$\sin(2A + A) \equiv \sin 2A\cos A + \cos 2A\sin A$

So $\sin 3A \equiv (2\sin A\cos A)\cos A + (\cos^2 A - \sin^2 A)\sin A$

$\equiv 2\sin A\cos^2 A + \cos^2 A\sin A - \sin^3 A$

$\equiv 3\sin A\cos^2 A - \sin^3 A$ *

$\equiv 3\sin A(1 - \sin^2 A) - \sin^3 A$

$\equiv 3\sin A - 4\sin^3 A$

$\cos 2A = \cos^2 A - \sin^2 A$ has been used here because the line marked * is a useful result when you need to find the formula for $\tan 3A$.

See Exercise 7C Questions 12 and 13 for similar expansions of $\cos 3A$ and $\tan 3A$.

Example 14

Given that $x = 3\sin\theta$ and $y = 3 - 4\cos 2\theta$, eliminate θ and express y in terms of x.

The equations can be rewritten as

$\sin\theta = \dfrac{x}{3} \quad \cos 2\theta = \dfrac{3 - y}{4}$

As $\cos 2\theta = 1 - 2\sin^2\theta$ for all values of θ,

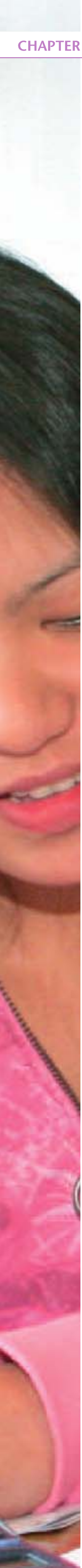
$\dfrac{3 - y}{4} = 1 - 2\left(\dfrac{x}{3}\right)^2$

So $\dfrac{y}{4} = 2\left(\dfrac{x}{3}\right)^2 - \frac{1}{4}$

or $y = 8\left(\dfrac{x}{3}\right)^2 - 1$

Be careful with this manipulation. Many errors occur in the early part of a solution.

This is the relationship: θ has been eliminated but the solution is not complete. Always make sure that you have answered the question: here you need to write $y = \ldots$.



Example 15

Solve $3\cos 2x - \cos x + 2 = 0$ for $0° \leqslant x \leqslant 360°$.

Using a double angle formula for $\cos 2x$

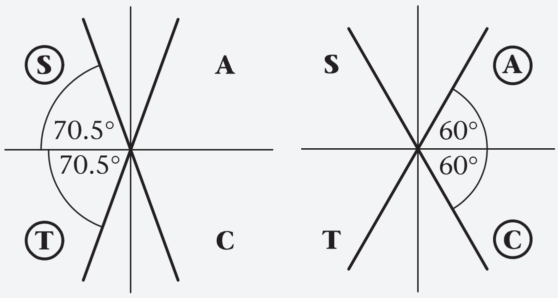
$$3\cos 2x - \cos x + 2 = 0$$

becomes

$$3(2\cos^2 x - 1) - \cos x + 2 = 0$$
$$6\cos^2 x - 3 - \cos x + 2 = 0$$
$$6\cos^2 x - \cos x - 1 = 0$$

So $(3\cos x + 1)(2\cos x - 1) = 0$

Solving: $\cos x = -\frac{1}{3}$ or $\cos x = \frac{1}{2}$



$\cos^{-1}\left(-\frac{1}{3}\right) = 109.5°$ $\cos^{-1}\left(\frac{1}{2}\right) = 60°$

So $x = 60°, 109.5°, 250.5°, 300°$

The term $\cos x$ in the equation dictates the choice you need to make; you need the form of $\cos 2x$ with $\cos x$ only, i.e. $\cos 2x \equiv 2\cos^2 x - 1$.

This quadratic equation factorises $6X^2 - X - 1 = (3X + 1)(2X - 1)$.

Solutions to $\cos x = -\frac{1}{3}$ are in the 2nd and 3rd quadrants. Solutions to $\cos x = \frac{1}{2}$ are in the 1st and 4th quadrants.

Remember that two solutions of $\cos x = k$ are $\cos^{-1} k$ and $360° - \cos^{-1} k$. In this case they all fall in the required interval.

Exercise 7C

In equations, give answers to 1 decimal place where appropriate.

1 Prove the following identities:

a $\dfrac{\cos 2A}{\cos A + \sin A} \equiv \cos A - \sin A$

b $\dfrac{\sin B}{\sin A} - \dfrac{\cos B}{\cos A} \equiv 2\operatorname{cosec} 2A \sin(B - A)$

c $\dfrac{1 - \cos 2\theta}{\sin 2\theta} \equiv \tan\theta$

d $\dfrac{\sec^2\theta}{1 - \tan^2\theta} \equiv \sec 2\theta$

e $2(\sin^3\theta\cos\theta + \cos^3\theta\sin\theta) \equiv \sin 2\theta$

f $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} \equiv 2$

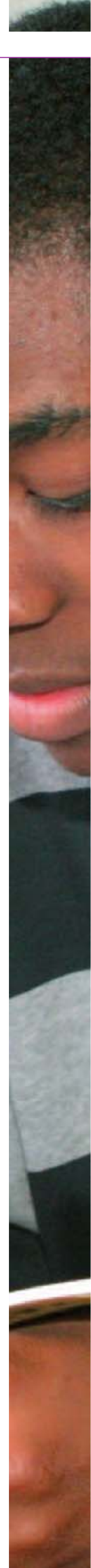
g $\operatorname{cosec}\theta - 2\cot 2\theta\cos\theta \equiv 2\sin\theta$

h $\dfrac{\sec\theta - 1}{\sec\theta + 1} \equiv \tan^2\dfrac{\theta}{2}$

i $\tan\left(\dfrac{\pi}{4} - x\right) \equiv \dfrac{1 - \sin 2x}{\cos 2x}$

2 **a** Show that $\tan\theta + \cot\theta \equiv 2\operatorname{cosec} 2\theta$.

b Hence find the value of $\tan 75° + \cot 75°$.



3 Solve the following equations, in the interval shown in brackets:

a $\sin 2\theta = \sin\theta \quad \{0 \leqslant \theta \leqslant 2\pi\}$

b $\cos 2\theta = 1 - \cos\theta \quad \{-180° < \theta \leqslant 180°\}$

c $3\cos 2\theta = 2\cos^2\theta \quad \{0 \leqslant \theta < 360°\}$

d $\sin 4\theta = \cos 2\theta \quad \{0 \leqslant \theta \leqslant \pi\}$

e $2\tan 2y \tan y = 3 \quad \{0 \leqslant y < 360°\}$

f $3\cos\theta - \sin\dfrac{\theta}{2} - 1 = 0 \quad \{0 \leqslant \theta < 720°\}$

g $\cos^2\theta - \sin 2\theta = \sin^2\theta \quad \{0 \leqslant \theta \leqslant \pi\}$

h $2\sin\theta = \sec\theta \quad \{0 \leqslant \theta \leqslant 2\pi\}$

i $2\sin 2\theta = 3\tan\theta \quad \{0 \leqslant \theta < 360°\}$

j $2\tan\theta = \sqrt{3}(1 - \tan\theta)(1 + \tan\theta) \quad \{0 \leqslant \theta \leqslant 2\pi\}$

k $5\sin 2\theta + 4\sin\theta = 0 \quad \{-180° < \theta \leqslant 180°\}$

l $\sin^2\theta = 2\sin 2\theta \quad \{-180° < \theta \leqslant 180°\}$

m $4\tan\theta = \tan 2\theta \quad \{0 \leqslant \theta < 360°\}$

4 Given that $p = 2\cos\theta$ and $q = \cos 2\theta$, express q in terms of p.

5 Eliminate θ from the following pairs of equations:

a $x = \cos^2\theta,\ y = 1 - \cos 2\theta$

b $x = \tan\theta,\ y = \cot 2\theta$

c $x = \sin\theta,\ y = \sin 2\theta$

d $x = 3\cos 2\theta + 1,\ y = 2\sin\theta$

6 **a** Prove that $(\cos 2\theta - \sin 2\theta)^2 \equiv 1 - \sin 4\theta$.

b Use the result to solve, for $0 \leqslant \theta < \pi$, the equation $\cos 2\theta - \sin 2\theta = \dfrac{1}{\sqrt{2}}$.
Give your answers in terms of π.

7 **a** Show that:

i $\sin\theta \equiv \dfrac{2\tan\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$ **ii** $\cos\theta \equiv \dfrac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$

b By writing the following equations as quadratics in $\tan\dfrac{\theta}{2}$, solve, in the interval $0 \leqslant \theta \leqslant 360°$:

i $\sin\theta + 2\cos\theta = 1$ **ii** $3\cos\theta - 4\sin\theta = 2$

8 **a** Using $\cos 2A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$, show that:

i $\cos^2\dfrac{x}{2} \equiv \dfrac{1 + \cos x}{2}$ **ii** $\sin^2\dfrac{x}{2} \equiv \dfrac{1 - \cos x}{2}$

These are known as the **half angle formulae**. (They are useful in integration.)

b Given that $\cos\theta = 0.6$, and that θ is acute, write down the values of:

i $\cos\dfrac{\theta}{2}$ **ii** $\sin\dfrac{\theta}{2}$ **iii** $\tan\dfrac{\theta}{2}$

c Show that $\cos^4\dfrac{A}{2} \equiv \frac{1}{8}(3 + 4\cos A + \cos 2A)$

9 **a** Show that $3\cos^2 x - \sin^2 x \equiv 1 + 2\cos 2x$.

b Hence sketch, for $-\pi \leqslant x \leqslant \pi$, the graph of $y = 3\cos^2 x - \sin^2 x$, showing the coordinates of points where the curve meets the axes.

10 **a** Express $2\cos^2\frac{\theta}{2} - 4\sin^2\frac{\theta}{2}$ in the form $a\cos\theta + b$, where a and b are constants.

b Hence solve $2\cos^2\frac{\theta}{2} - 4\sin^2\frac{\theta}{2} = -3$, in the interval $0 \leqslant \theta < 360°$.

11 **a** Use the identity $\sin^2 A + \cos^2 A \equiv 1$ to show that $\sin^4 A + \cos^4 A \equiv \frac{1}{2}(2 - \sin^2 2A)$.

b Deduce that $\sin^4 A + \cos^4 A \equiv \frac{1}{4}(3 + \cos 4A)$.

c Hence solve $8\sin^4\theta + 8\cos^4\theta = 7$, for $0 < \theta < \pi$.

12 **a** By expanding $\cos(2A + A)$ show that $\cos 3A \equiv 4\cos^3 A - 3\cos A$.

b Hence solve $8\cos^3\theta - 6\cos\theta - 1 = 0$, for $\{0 \leqslant \theta \leqslant 360°\}$.

13 **a** Show that $\tan 3\theta \equiv \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

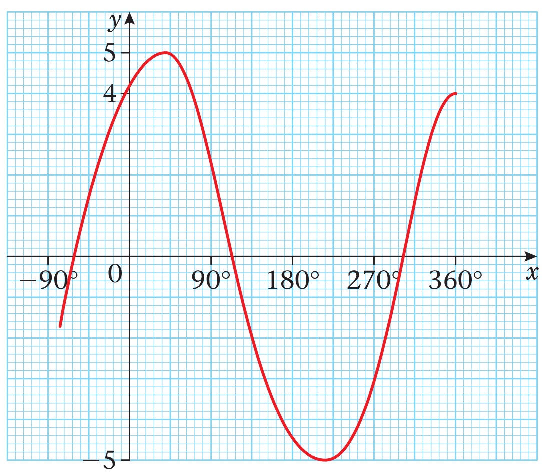
b Given that θ is acute such that $\cos\theta = \frac{1}{3}$, show that

$\tan 3\theta = \dfrac{10\sqrt{2}}{23}$.

Hint: Divide formulae for $\sin 3\theta$ and $\cos 3\theta$. See Example 13 for a useful form of $\sin 3\theta$, and use a similar form for $\cos 3\theta$.

7.4 You can write expressions of the form $a\cos\theta + b\sin\theta$, where a and b are constants, as a sine function only or a cosine function only.

If you sketch, or draw on your calculator, the graph of $y = 3\sin x + 4\cos x$ you will see that it has the form of $y = \sin x$ or $y = \cos x$ but stretched vertically and translated horizontally.



If you draw the graph of $y = 5\sin(x + \tan^{-1}\{\frac{4}{3}\})$ or $y = 5\cos(x - \tan^{-1}\{\frac{3}{4}\})$, you will see that they are the same as $y = 3\sin x + 4\cos x$.

Using the addition formulae you can show that all expressions of the form $a\cos\theta + b\sin\theta$ can be expressed in one of the forms
$R\sin(x \pm \alpha)$ where $R > 0$, and $0 < \alpha < 90°$, or
$R\cos(x \pm \beta)$ where $R > 0$, and $0 < \beta < 90°$.

Remember: the graph of $y = a\,f(x - \alpha)$ is the graph of $y = f(x)$ stretched vertically by a factor of a and translated horizontally by α.

Example 16

Show that you can express $3\sin x + 4\cos x$ in the form $R\sin(x + \alpha)$, where $R > 0$, $0 < \alpha < 90°$, giving your values of R and α to 1 decimal place where appropriate.

$R\sin(x + \alpha) \equiv R\sin x\cos\alpha + R\cos x\sin\alpha$

Use $\sin(A + B) \equiv \sin A\cos B + \cos A\sin B$, and multiply through by R.

Let $3\sin x + 4\cos x \equiv R\sin x\cos\alpha + R\cos x\sin\alpha$

For this to be true for all values of x, the coefficients of $\sin x$ and $\cos x$ on both sides of the identity have to be equal.

So $R\cos\alpha = 3$ and $R\sin\alpha = 4$

Equations of this sort can always be solved, and so R and α can always be found.

Divide the equations to find $\tan\alpha$

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{4}{3} \text{ or } \tan\alpha = \frac{4}{3}$$

So $\alpha = 53.1°$ (1 d.p.)

Square and add the equations to find R^2:

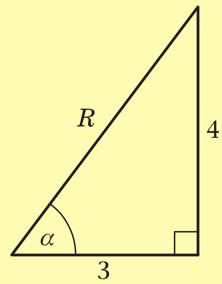
$$R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 4^2$$

So $R^2(\cos^2\alpha + \sin^2\alpha) = 3^2 + 4^2$

So $R = \sqrt{3^2 + 4^2} = 5$

$3\sin x + 4\cos x \equiv 5\sin(x + 53.1°)$

You could draw a right-angled triangle with $\cos\alpha = \frac{3}{R}$ and $\sin\alpha = \frac{4}{R}$



So $\tan\alpha = \frac{4}{3}$ and
$R^2 = 3^2 + 4^2 \Rightarrow R = 5$.

You could equally have shown that $3\sin x + 4\cos x \equiv 5\cos(x - 36.9°)$ by setting $3\sin x + 4\cos x \equiv R\cos(x - \alpha)$ and solving for R and α, as in the example.

Example 17

a Show that you can express $\sin x - \sqrt{3}\cos x$ in the form $R\sin(x - \alpha)$, where $R > 0$, $0 < \alpha < \frac{\pi}{2}$.

b Hence sketch the graph of $y = \sin x - \sqrt{3}\cos x$.

a Set $\sin x - \sqrt{3}\cos x \equiv R\sin(x - \alpha)$

$\sin x - \sqrt{3}\cos x \equiv R\sin x\cos\alpha - R\cos x\sin\alpha$

Expand $\sin(x - \alpha)$ and multiply by R.

So $R\cos\alpha = 1$ and $R\sin\alpha = \sqrt{3}$

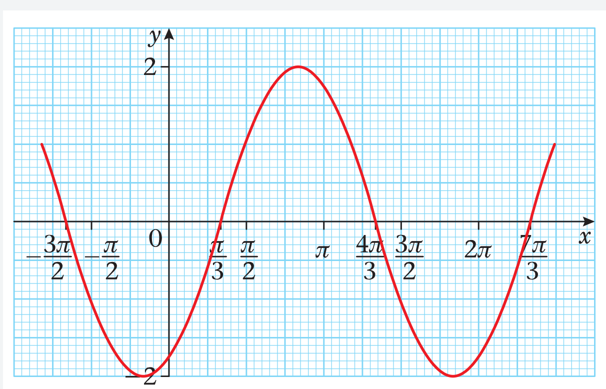
Compare the coefficients of $\sin x$ and $\cos x$ on both sides of the identity.

Dividing, $\tan\alpha = \sqrt{3}$, so $\alpha = \frac{\pi}{3}$

Squaring and adding: $R = 2$

So $\sin x - \sqrt{3}\cos x \equiv 2\sin\left(x - \frac{\pi}{3}\right)$

b $y = \sin x - \sqrt{3}\cos x \equiv 2\sin\left(x - \frac{\pi}{3}\right)$



You can sketch $y = 2\sin\left(x - \frac{\pi}{3}\right)$ by translating $y = \sin x$ by $\frac{\pi}{3}$ to the right and then stretching by a scale factor of 2 in the y-direction.

Example 18

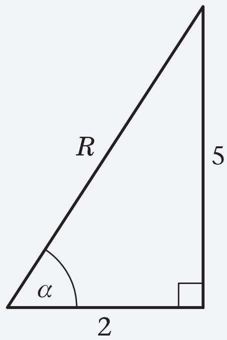
a Express $2\cos\theta + 5\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$, $0 < \alpha < 90°$.

b Hence solve, for $0 < \theta < 360°$, the equation $2\cos\theta + 5\sin\theta = 3$.

a Set $2\cos\theta + 5\sin\theta \equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

So $R\cos\alpha = 2$ and $R\sin\alpha = 5$

> Compare the coefficients of $\sin x$ and $\cos x$ on both sides of the identity.



> Draw a right-angled triangle with $\cos\alpha = \frac{2}{R}$ and $\sin\alpha = \frac{5}{R}$

$\therefore$ $\tan\alpha = 2.5$ and $R = \sqrt{2^2 + 5^2} = \sqrt{29}$

So $2\cos\theta + 5\sin\theta \equiv \sqrt{29}\cos(\theta - 68.2°)$

b The solutions of $2\cos\theta + 5\sin\theta = 3$ are the same as those of $\sqrt{29}\cos(\theta - 68.2°) = 3$.

Divide the equation by $\sqrt{29}$.

So $\cos(\theta - 68.2°) = \frac{3}{\sqrt{29}}$

$\cos^{-1}\left(\frac{3}{\sqrt{29}}\right) = 56.1\ldots°$

> As $0 < \theta < 360°$, the interval for $(\theta - 68.2°)$ is $-68.2° < (\theta - 68.2°) < 291.8°$.
>
> $\frac{3}{\sqrt{29}}$ is +ve, so solutions for $\theta - 68.2$ are in the 1st and 4th quadrants.

So $\theta - 68.2° = -56.1\ldots°, 56.1\ldots°$

$\theta = 12.1°, 124.3°$ (to the nearest 0.1°)

Example 19

Without using calculus, find the maximum value of $12\cos\theta + 5\sin\theta$, and give the smallest positive value of θ at which it arises.

Set $12\cos\theta + 5\sin\theta \equiv R\cos(\theta - \alpha)$

> The most convenient forms to use here are $R\sin(\theta + \alpha)$ or $R\cos(\theta - \alpha)$ as the sign in the expanded form is the same as that in $12\cos\theta + 5\sin\theta$. If the signs do not match up, it will be more difficult for you.

So $12\cos\theta + 5\sin\theta \equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

So $R\cos\alpha = 12$ and $R\sin\alpha = 5$

$R = 13$ and $\tan\alpha = \frac{5}{12} \Rightarrow \alpha = 22.6°$

So $12\cos\theta + 5\sin\theta \equiv 13\cos(\theta - 22.6°)$

The maximum value of $13\cos(\theta - 22.6°)$ is 13 and occurs when $\cos(\theta - 22.6°) = 1$;
i.e. when $\theta - 22.6° = \ldots, -360°, 0°, 360°, \ldots$
The smallest positive value of θ, therefore, is 22.6°.

- **For positive values of a and b,**
  $a\sin\theta \pm b\cos\theta$ can be expressed in the form $R\sin(\theta \pm \alpha)$, with $R > 0$ and $0 < \alpha < 90°$ $\left(\text{or } \frac{\pi}{2}\right)$

  $a\cos\theta \pm b\sin\theta$ can be expressed in the form $R\cos(\theta \mp \alpha)$, with $R > 0$ and $0 < \alpha < 90°$ $\left(\text{or } \frac{\pi}{2}\right)$

  where $R\cos\alpha = a$ and $R\sin\alpha = b$

  and $R = \sqrt{a^2 + b^2}$.

Do not quote these results, but they are useful check points.

Note: When solving equations of the form $a\cos\theta + b\sin\theta = c$, use the '$R$ formula', unless $c = 0$, when the equation reduces to $\tan\theta = k$.

Exercise 7D

Give all angles to the nearest 0.1° and non-exact values of R in surd form.

1 Given that $5\sin\theta + 12\cos\theta \equiv R\sin(\theta + \alpha)$, find the value of R, $R > 0$, and the value of $\tan\alpha$.

2 Given that $\sqrt{3}\sin\theta + \sqrt{6}\cos\theta \equiv 3\cos(\theta - \alpha)$, where $0 < \alpha < 90°$, find the value of α.

3 Given that $2\sin\theta - \sqrt{5}\cos\theta \equiv -3\cos(\theta + \alpha)$, where $0 < \alpha < 90°$, find the value of α.

4 Show that:

a $\cos\theta + \sin\theta \equiv \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$

b $\sqrt{3}\sin 2\theta - \cos 2\theta \equiv 2\sin\left(2\theta - \frac{\pi}{6}\right)$

5 Prove that $\cos 2\theta - \sqrt{3}\sin 2\theta \equiv 2\cos\left(2\theta + \frac{\pi}{3}\right) \equiv -2\sin\left(2\theta - \frac{\pi}{6}\right)$.

6 Find the value of R, where $R > 0$, and the value of α, where $0 < \alpha < 90°$, in each of the following cases:

a $\sin\theta + 3\cos\theta \equiv R\sin(\theta + \alpha)$

b $3\sin\theta - 4\cos\theta \equiv R\sin(\theta - \alpha)$

c $2\cos\theta + 7\sin\theta \equiv R\cos(\theta - \alpha)$

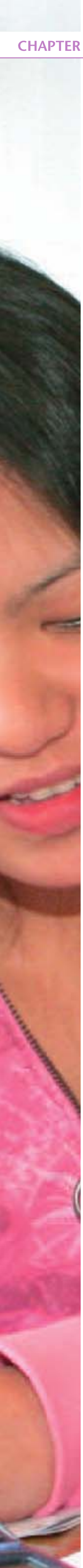
d $\cos 2\theta - 2\sin 2\theta \equiv R\cos(2\theta + \alpha)$

7 **a** Show that $\cos\theta - \sqrt{3}\sin\theta$ can be written in the form $R\cos(\theta + \alpha)$, with $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

b Hence sketch the graph of $y = \cos\theta - \sqrt{3}\sin\theta$, $0 < \alpha < 2\pi$, giving the coordinates of points of intersection with the axes.

8 **a** Show that $3\sin 3\theta - 4\cos 3\theta$ can be written in the form $R\sin(3\theta - \alpha)$, with $R > 0$ and $0 < \alpha < 90°$.

b Deduce the minimum value of $3\sin 3\theta - 4\cos 3\theta$ and work out the smallest positive value of θ at which it occurs.



9 **a** Show that $\cos 2\theta + \sin 2\theta$ can be written in the form $R\sin(2\theta + \alpha)$, with $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

b Hence solve, in the interval $0 \leqslant \theta < 2\pi$, the equation $\cos 2\theta + \sin 2\theta = 1$, giving your answers as rational multiples of π.

10 **a** Express $7\cos\theta - 24\sin\theta$ in the form $R\cos(\theta + \alpha)$, with $R > 0$ and $0 < \alpha < 90°$.

b The graph of $y = 7\cos\theta - 24\sin\theta$ meets the y-axis at P. State the coordinates of P.

c Write down the maximum and minimum values of $7\cos\theta - 24\sin\theta$.

d Deduce the number of solutions, in the interval $0 < \theta < 360°$, of the following equations:

i $7\cos\theta - 24\sin\theta = 15$ **ii** $7\cos\theta - 24\sin\theta = 26$ **iii** $7\cos\theta - 24\sin\theta = -25$

11 **a** Express $5\sin^2\theta - 3\cos^2\theta + 6\sin\theta\cos\theta$ in the form $a\sin 2\theta + b\cos 2\theta + c$, where a, b and c are constants.

b Hence find the maximum and minimum values of $5\sin^2\theta - 3\cos^2\theta + 6\sin\theta\cos\theta$.

12 Solve the following equations, in the interval given in brackets:

a $6\sin x + 8\cos x = 5\sqrt{3}$ [0, 360°] **b** $2\cos 3\theta - 3\sin 3\theta = -1$ [0, 90°]

c $8\cos\theta + 15\sin\theta = 10$ [0, 360°] **d** $5\sin\frac{x}{2} - 12\cos\frac{x}{2} = -6.5$ [−360°, 360°]

13 Solve the following equations, in the interval given in brackets:

a $\sin x\cos x = 1 - 2.5\cos 2x$ [0, 360°] **b** $\cot\theta + 2 = \operatorname{cosec}\theta$ [$0 < \theta < 360°$, $\theta \neq 180°$]

c $\sin\theta = 2\cos\theta - \sec\theta$ [0, 180°] **d** $\sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right) + (\sqrt{3} - 1)\sin\theta = 2$ [0, 2π]

14 Solve, if possible, in the interval $0 < \theta < 360°$, $\theta \neq 180°$, the equation $\dfrac{4 - 2\sqrt{2}\sin\theta}{1 + \cos\theta} = k$ in the case when k is equal to:

a 4 **b** 2 **c** 1 **d** 0 **e** −1

15 A class were asked to solve $3\cos\theta = 2 - \sin\theta$ for $0 \leqslant \theta \leqslant 360°$. One student expressed the equation in the form $R\cos(\theta - \alpha) = 2$, with $R > 0$ and $0 < \alpha < 90°$, and correctly solved the equation.

a Find the values of R and α and hence find her solutions.

Another student decided to square both sides of the equation and then form a quadratic equation in $\sin\theta$.

b Show that the correct quadratic equation is $10\sin^2\theta - 4\sin\theta - 5 = 0$.

c Solve this equation, for $0 \leqslant \theta < 360°$.

d Explain why not all of the answers satisfy $3\cos\theta = 2 - \sin\theta$.

7.5 You can express sums and differences of sines and cosines as products of sines and/or cosines by using the 'factor formulae'.

- $\sin P + \sin Q \equiv 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$
- $\cos P + \cos Q \equiv 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$
- $\sin P - \sin Q \equiv 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$
- $\cos P - \cos Q \equiv -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$

These identities are derived from the addition formulae.

Example 20

Use the formulae for $\sin(A + B)$ and $\sin(A - B)$ to derive the result that

$$\sin P + \sin Q \equiv 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$$

$\sin(A + B) \equiv \sin A\cos B + \cos A\sin B$

and $\sin(A - B) \equiv \sin A\cos B - \cos A\sin B$

Add the two identities:

$\sin(A + B) + \sin(A - B) \equiv 2\sin A\cos B$

Let $A + B = P$ and $A - B = Q$,

then $A = \dfrac{P+Q}{2}$ and $B = \dfrac{P-Q}{2}$

$\sin P + \sin Q \equiv 2\sin\left(\dfrac{P+Q}{2}\right)\cos\left(\dfrac{P-Q}{2}\right)$

The other three factor formulae are proved in a similar manner, by adding or subtracting two appropriate addition formulae.
See Exercise 7E.

This result is useful in integration, e.g.

$\int 2\sin 4x\cos x\,dx = \int(\sin 5x + \sin 3x)dx.$

Example 21

Using the result that $\sin P - \sin Q \equiv 2\cos\left(\dfrac{P+Q}{2}\right)\sin\left(\dfrac{P-Q}{2}\right)$

a show that $\sin 105° - \sin 15° = \dfrac{1}{\sqrt{2}}$

b solve, for $0 \leqslant \theta \leqslant \pi$, $\sin 4\theta - \sin 3\theta = 0$

a $\sin 105° - \sin 15°$

$= 2\cos\left(\dfrac{105° + 15°}{2}\right)\sin\left(\dfrac{105° - 15°}{2}\right)$

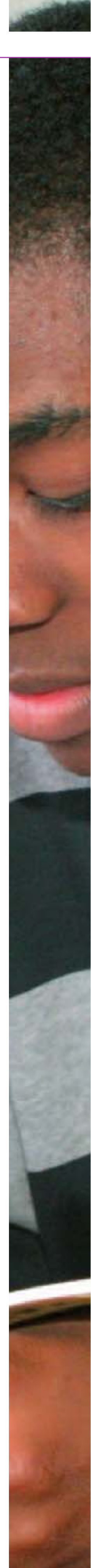
$= 2\cos 60°\sin 45°$

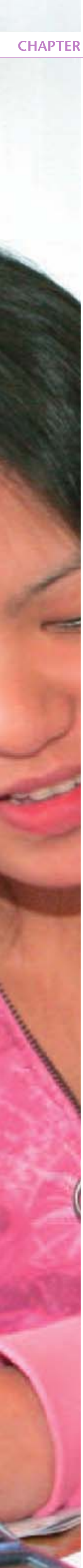
$= 2\left(\dfrac{1}{2}\right)\left(\dfrac{1}{\sqrt{2}}\right)$

$= \dfrac{1}{\sqrt{2}}$

Let $P = 105°$ and $Q = 15°$.

Remember: $\cos 60° = \frac{1}{2}$, $\sin 45° = \dfrac{1}{\sqrt{2}}$.





b $\sin 4\theta - \sin 3\theta = 2\cos\left(\frac{7\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)$

Let $P = 4\theta$ and $Q = 3\theta$.

The solutions of $2\cos\left(\frac{7\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) = 0$ are either

$$\cos\left(\frac{7\theta}{2}\right) = 0$$

As $0 \leqslant \theta \leqslant \pi$, the interval for $\frac{7\theta}{2}$ is $0 \leqslant \frac{7\theta}{2} \leqslant \frac{7\pi}{2}$.

so $\frac{7\theta}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$\therefore \quad \theta = \frac{\pi}{7}, \frac{3\pi}{7}, \frac{5\pi}{7}, \pi$

or $\sin\left(\frac{\theta}{2}\right) = 0$

The interval for $\frac{\theta}{2}$ is $0 \leqslant \frac{\theta}{2} \leqslant \frac{\pi}{2}$.

so $\frac{\theta}{2} = 0 \quad \therefore \quad \theta = 0$

Answers are $\theta = 0, \frac{\pi}{7}, \frac{3\pi}{7}, \frac{5\pi}{7}, \pi$

Example 22

Prove that $\dfrac{\sin(x + 2y) + \sin(x + y) + \sin x}{\cos(x + 2y) + \cos(x + y) + \cos x} \equiv \tan(x + y)$.

In the numerator

$\sin(x + 2y) + \sin x$

Use $\sin P + \sin Q \equiv 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$ with $P = x + 2y$ and $Q = x$.

$\equiv 2\sin\left(\frac{2x+2y}{2}\right)\cos\left(\frac{2y}{2}\right)$

$\equiv 2\sin(x + y)\cos y$

So $\sin(x + 2y) + \sin(x + y) + \sin x$

$\equiv \sin(x + y) + 2\sin(x + y)\cos y$

$\equiv \sin(x + y)(1 + 2\cos y)$ ①

Factorise.

Similarly for the denominator

$\cos(x + 2y) + \cos(x + y) + \cos x$

Use $\cos P + \cos Q \equiv 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$ with $P = x + 2y$ and $Q = x$.

$\equiv \cos(x + y) + 2\cos(x + y)\cos y$

$\equiv \cos(x + y)(1 + 2\cos y)$ ②

Factorise.

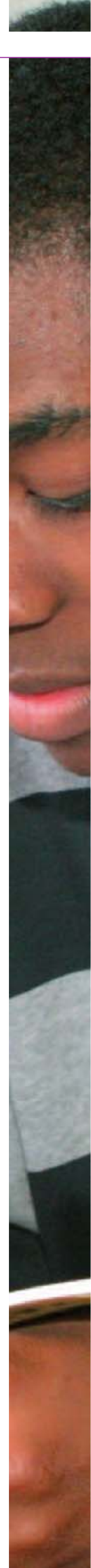
so $\dfrac{\sin(x + 2y) + \sin(x + y) + \sin x}{\cos(x + 2y) + \cos(x + y) + \cos x}$

Use results ① and ②.

$\equiv \dfrac{\sin(x + y)(1 + 2\cos y)}{\cos(x + y)(1 + 2\cos y)}$

Cancel.

$\equiv \tan(x + y)$



Exercise 7E

1 **a** Show that $\sin(A+B) + \sin(A-B) \equiv 2\sin A\cos B$.

b Deduce that $\sin P + \sin Q \equiv 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$.

c Use part **a** to express the following as the sum of two sines:

i $2\sin 7\theta\cos 2\theta$ **ii** $2\sin 12\theta\cos 5\theta$

d Use the result in **b** to solve, in the interval $0 \leqslant \theta \leqslant 180°$, $\sin 3\theta + \sin\theta = 0$.

e Prove that $\dfrac{\sin 7\theta + \sin\theta}{\sin 5\theta + \sin 3\theta} \equiv \dfrac{\cos 3\theta}{\cos\theta}$.

2 **a** Show that $\sin(A+B) - \sin(A-B) \equiv 2\cos A\sin B$.

b Express the following as the difference of two sines:

i $2\cos 5x\sin 3x$ **ii** $\cos 2x\sin x$ **iii** $6\cos\frac{3}{2}x\sin\frac{1}{2}x$

c Using the result in **a** show that $\sin P - \sin Q \equiv 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$.

d Deduce that $\sin 56° - \sin 34° = \sqrt{2}\sin 11°$.

3 **a** Show that $\cos(A+B) + \cos(A-B) \equiv 2\cos A\cos B$.

b Express as a sum of cosines **i** $2\cos\dfrac{5\theta}{2}\cos\dfrac{\theta}{2}$ **ii** $5\cos 2x\cos 3x$

c Show that $\cos P + \cos Q \equiv 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$.

d Prove that $\dfrac{\sin 3\theta - \sin\theta}{\cos 3\theta + \cos\theta} \equiv \tan\theta$.

4 **a** Show that $\cos(A+B) - \cos(A-B) \equiv -2\sin A\sin B$.

b Hence show that $\cos P - \cos Q \equiv -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$.

c Deduce that $\cos 2\theta - 1 \equiv -2\sin^2\theta$.

d Solve, in the interval $0 \leqslant \theta \leqslant 180°$, $\cos 3\theta + \sin 2\theta - \cos\theta = 0$.

Hint to Question 4c: What is the value of $\cos 0°$?

5 Express the following as a sum or difference of sines or cosines:

a $2\sin 8x\cos 2x$ **b** $\cos 5x\cos x$ **c** $3\sin x\sin 7x$

d $\cos 100°\cos 40°$ **e** $10\cos\dfrac{3x}{2}\sin\dfrac{x}{2}$ **f** $2\sin 30°\cos 10°$

6 Show, without using a calculator, that $2\sin 82\frac{1}{2}°\cos 37\frac{1}{2}° = \frac{1}{2}(\sqrt{3}+\sqrt{2})$.

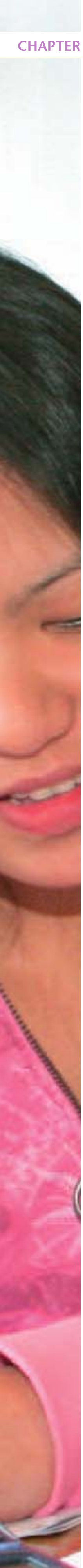
7 Express, in their simplest form, as a product of sines and/or cosines:

a $\sin 12x + \sin 8x$ **b** $\cos(x+2y) - \cos(2y-x)$ **c** $(\cos 4x + \cos 2x)\sin x$

d $\sin 95° - \sin 5°$ **e** $\cos\dfrac{\pi}{15} + \cos\dfrac{\pi}{12}$ **f** $\sin 150° + \sin 20°$

8 Using the identity $\cos P + \cos Q \equiv 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$, show that

$$\cos\theta + \cos\left(\theta + \frac{2\pi}{3}\right) + \cos\left(\theta + \frac{4\pi}{3}\right) = 0.$$



9 Prove that $\dfrac{\sin 75° + \sin 15°}{\cos 15° - \cos 75°} = \sqrt{3}$.

10 Solve the following equations:

a $\cos 4x = \cos 2x$, for $0 \leqslant x \leqslant 180°$

b $\sin 3\theta - \sin\theta = 0$, for $0 \leqslant \theta \leqslant 2\pi$

c $\sin(x + 20°) + \sin(x - 10°) = \cos 15°$, for $0 \leqslant x \leqslant 360°$

d $\sin 3\theta - \sin\theta = \cos 2\theta$, for $0 \leqslant \theta \leqslant 2\pi$

11 Prove the identities

a $\dfrac{\sin 7\theta - \sin 3\theta}{\sin\theta\cos\theta} \equiv 4\cos 5\theta$ **b** $\dfrac{\cos 2\theta + \cos 4\theta}{\sin 2\theta - \sin 4\theta} \equiv -\cot\theta$

c $\sin^2(x + y) - \sin^2(x - y) \equiv \sin 2x \sin 2y$ **d** $\cos x + 2\cos 3x + \cos 5x \equiv 4\cos^2 x \cos 3x$

12 **a** Prove that $\cos\theta + \sin 2\theta - \cos 3\theta \equiv \sin 2\theta(1 + 2\sin\theta)$.

b Hence solve, for $0 \leqslant \theta \leqslant 2\pi$, $\cos\theta + \sin 2\theta = \cos 3\theta$.

Mixed exercise 7F

1 The lines l_1 and l_2, with equations $y = 2x$ and $3y = x - 1$ respectively, are drawn on the same set of axes. Given that the scales are the same on both axes and that the angles that l_1 and l_2 make with the positive x-axis are A and B respectively,

a write down the value of $\tan A$ and the value of $\tan B$;

b without using your calculator, work out the acute angle between l_1 and l_2.

2 Given that $\sin x = \dfrac{1}{\sqrt{5}}$ where x is acute, and that $\cos(x - y) = \sin y$, show that $\tan y = \dfrac{\sqrt{5} + 1}{2}$.

3 Using $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$ with an appropriate value of θ,

a show that $\tan\dfrac{\pi}{8} = \sqrt{2} - 1$.

b Use the result in **a** to find the exact value of $\tan\dfrac{3\pi}{8}$.

4 In △ABC, AB = 5 cm and AC = 4 cm, ∠ABC = $(\theta - 30)°$ and ∠ACB = $(\theta + 30)°$. Using the sine rule, show that $\tan\theta = 3\sqrt{3}$.

5 Two of the angles, A and B, in △ABC are such that $\tan A = \frac{3}{4}$, $\tan B = \frac{5}{12}$.

a Find the exact value of **i** $\sin(A + B)$ **ii** $\tan 2B$

b By writing C as $180° - (A + B)$, show that $\cos C = -\frac{33}{65}$.

6 Show that

a $\sec\theta\,\mathrm{cosec}\,\theta \equiv 2\,\mathrm{cosec}\,2\theta$ **b** $\dfrac{1 - \cos 2x}{1 + \cos 2x} \equiv \sec^2 x - 1$

c $\cot\theta - 2\cot 2\theta \equiv \tan\theta$ **d** $\cos^4 2\theta - \sin^4 2\theta \equiv \cos 4\theta$

e $\tan\left(\dfrac{\pi}{4} + x\right) - \tan\left(\dfrac{\pi}{4} - x\right) \equiv 2\tan 2x$ **f** $\sin(x + y)\sin(x - y) \equiv \cos^2 y - \cos^2 x$

g $1 + 2\cos 2\theta + \cos 4\theta \equiv 4\cos^2\theta\cos 2\theta$

7 The angles x and y are acute angles such that $\sin x = \frac{2}{\sqrt{5}}$ and $\cos y = \frac{3}{\sqrt{10}}$.

a Show that $\cos 2x = -\frac{3}{5}$.

b Find the value of $\cos 2y$.

c Show without using your calculator, that

i $\tan(x + y) = 7$ **ii** $x - y = \frac{\pi}{4}$

8 Given that $\sin x \cos y = \frac{1}{2}$ and $\cos x \sin y = \frac{1}{3}$,

a show that $\sin(x + y) = 5\sin(x - y)$.

Given also that $\tan y = k$, express in terms of k:

b $\tan x$

c $\tan 2x$

9 Solve the following equations in the interval given in brackets:

a $\sqrt{3}\sin 2\theta + 2\sin^2\theta = 1$ $\{0 \leqslant \theta \leqslant \pi\}$

b $\sin 3\theta \cos 2\theta = \sin 2\theta \cos 3\theta$ $\{0 \leqslant \theta \leqslant 2\pi\}$

c $\sin(\theta + 40°) + \sin(\theta + 50°) = 0$ $\{0 \leqslant \theta \leqslant 360°)$

d $\sin^2\frac{\theta}{2} = 2\sin\theta$ $\{0 \leqslant \theta \leqslant 360°\}$

e $2\sin\theta = 1 + 3\cos\theta$ $\{0 \leqslant \theta \leqslant 360°\}$

f $\cos 5\theta = \cos 3\theta$ $\{0 \leqslant \theta \leqslant \pi\}$

g $\cos 2\theta = 5\sin\theta$ $\{-\pi \leqslant \theta \leqslant \pi\}$.

10 The first three terms of an arithmetic series are $\sqrt{3}\cos\theta$, $\sin(\theta - 30°)$ and $\sin\theta$, where θ is acute. Find the value of θ.

11 Solve, for $0 \leqslant \theta \leqslant 360°$, $\cos(\theta + 40°)\cos(\theta - 10°) = 0.5$.

12 Without using calculus, find the maximum and minimum value of the following expressions. In each case give the smallest positive value of θ at which each occurs.

a $\sin\theta\cos 10° - \cos\theta\sin 10°$

b $\cos 30°\cos\theta - \sin 30°\sin\theta$

c $\sin\theta + \cos\theta$

13 **a** Express $\sin x - \sqrt{3}\cos x$ in the form $R\sin(x - \alpha)$, with $R > 0$ and $0 < \alpha < 90°$.

b Hence sketch the graph of $y = \sin x - \sqrt{3}\cos x$ $\{-360° \leqslant x \leqslant 360°\}$, giving the coordinates of all points of intersection with the axes.

14 Given that $7\cos 2\theta + 24\sin 2\theta \equiv R\cos(2\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$, find:

a the value of R and the value of α, to 2 decimal places

b the maximum value of $14\cos^2\theta + 48\sin\theta\cos\theta$

15 **a** Given that α is acute and $\tan\alpha = \frac{3}{4}$, prove that

$$3\sin(\theta + \alpha) + 4\cos(\theta + \alpha) \equiv 5\cos\theta$$

b Given that $\sin x = 0.6$ and $\cos x = -0.8$, evaluate $\cos(x + 270)°$ and $\cos(x + 540)°$. E

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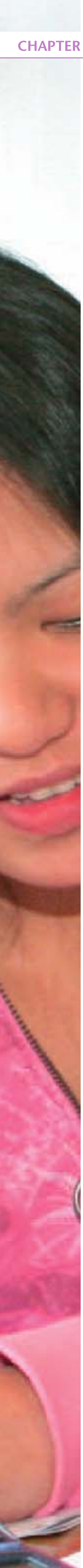

16 **a** Without using a calculator, find the values of:

i $\sin 40° \cos 10° - \cos 40° \sin 10°$ **ii** $\frac{1}{\sqrt{2}} \cos 15° - \frac{1}{\sqrt{2}} \sin 15°$ **iii** $\frac{1 - \tan 15°}{1 + \tan 15°}$

b Find, to 1 decimal place, the values of x, $0 \leqslant x \leqslant 360°$, which satisfy the equation

$2 \sin x = \cos(x - 60)$

E

17 **a** Prove, by counter example, that the statement

'$\sec(A + B) \equiv \sec A + \sec B$, for all A and B'

is false.

b Prove that $\tan \theta + \cot \theta \equiv 2 \operatorname{cosec} 2\theta$, $\theta \neq \frac{n\pi}{2}$, $n \in \mathbb{Z}$.

E

18 Using the formula $\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$:

a Show that $\cos(A - B) - \cos(A + B) \equiv 2 \sin A \sin B$.

b Hence show that $\cos 2x - \cos 4x \equiv 2 \sin 3x \sin x$.

c Find all solutions in the range $0 \leqslant x \leqslant \pi$ of the equation

$\cos 2x - \cos 4x = \sin x$

giving all your solutions in multiples of π radians.

E

19 **a** Given that $\cos(x + 30°) = 3 \cos(x - 30°)$, prove that $\tan x = -\frac{\sqrt{3}}{2}$.

b **i** Prove that $\frac{1 - \cos 2\theta}{\sin 2\theta} = \tan \theta$.

ii Verify that $\theta = 180°$ is a solution of the equation $\sin 2\theta = 2 - 2 \cos 2\theta$.

iii Using the result in part **i**, or otherwise, find the two other solutions, $0 < \theta < 360°$, of the equation $\sin 2\theta = 2 - 2 \cos 2\theta$.

E

20 **a** Express $1.5 \sin 2x + 2 \cos 2x$ in the form $R \sin(2x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$, giving your values of R and α to 3 decimal places where appropriate.

b Express $3 \sin x \cos x + 4 \cos^2 x$ in the form $a \sin 2x + b \cos 2x + c$, where a, b and c are constants to be found.

c Hence, using your answer to part **a**, deduce the maximum value of $3 \sin x \cos x + 4 \cos^2 x$.

E

Summary of key points

1 The addition (or compound angle) formulae are

- $\sin(A + B) \equiv \sin A \cos B + \cos A \sin B$ $\quad$ $\sin(A - B) \equiv \sin A \cos B - \cos A \sin B$
- $\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$ $\quad$ $\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$
- $\tan(A + B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ $\quad$ $\tan(A - B) \equiv \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

2 The double angle formulae are

- $\sin 2A \equiv 2 \sin A \cos A$
- $\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$
- $\tan 2A \equiv \dfrac{2 \tan A}{1 - \tan^2 A}$

3 Expressions of the form $a \sin\theta + b \cos\theta$ can be rewritten in terms of a sine only or a cosine only, as follows:

For positive values of a and b,

$a \sin\theta \pm b \cos\theta \equiv R \sin(\theta \pm \alpha)$, with $R > 0$ and $0 < \alpha < 90°$,
$a \cos\theta \pm b \sin\theta \equiv R \cos(\theta \mp \alpha)$, with $R > 0$ and $0 < \alpha < 90°$

where $R \cos\alpha = a$, $R \sin\alpha = b$ and $R = \sqrt{a^2 + b^2}$.

Remember you can always use 'the R formula' to solve equations of the form $a \cos\theta + b \sin\theta = c$, where a, b and c are constants, but if $c = 0$, the equation reduces to the form $\tan\theta = k$.

4 Products of sines and/or cosines can be expressed as the sum or difference of sines or cosines, using the formulae:

$2 \sin A \cos B \equiv \sin(A + B) + \sin(A - B)$ $\quad$ $2 \cos A \cos B \equiv \cos(A + B) + \cos(A - B)$

$2 \cos A \sin B \equiv \sin(A + B) - \sin(A - B)$ $\quad$ $2 \sin A \sin B \equiv -[\cos(A + B) - \cos(A - B)]$

5 Sums or differences of sines or cosines can be expressed as a product of sines and/or cosines, using 'the factor formulae':

$\sin P + \sin Q \equiv 2 \sin\left(\dfrac{P + Q}{2}\right)\cos\left(\dfrac{P - Q}{2}\right)$ $\quad$ $\cos P + \cos Q \equiv 2 \cos\left(\dfrac{P + Q}{2}\right)\cos\left(\dfrac{P - Q}{2}\right)$

$\sin P - \sin Q \equiv 2 \cos\left(\dfrac{P + Q}{2}\right)\sin\left(\dfrac{P - Q}{2}\right)$ $\quad$ $\cos P - \cos Q \equiv -2 \sin\left(\dfrac{P + Q}{2}\right)\sin\left(\dfrac{P - Q}{2}\right)$