C4 INTEGRATION — Answers - Worksheet C

1 **a** $\dfrac{3x+5}{(x+1)(x+3)} \equiv \dfrac{A}{x+1} + \dfrac{B}{x+3}$

$3x + 5 \equiv A(x + 3) + B(x + 1)$

$x = -1 \Rightarrow 2 = 2A \Rightarrow A = 1$

$x = -3 \Rightarrow -4 = -2B \Rightarrow B = 2$

$\therefore \dfrac{3x+5}{(x+1)(x+3)} \equiv \dfrac{1}{x+1} + \dfrac{2}{x+3}$

b $= \int (\dfrac{1}{x+1} + \dfrac{2}{x+3})\ dx$

$= \ln|x + 1| + 2\ln|x + 3| + c$

2 $\dfrac{3}{(t-2)(t+1)} \equiv \dfrac{A}{t-2} + \dfrac{B}{t+1}$

$3 \equiv A(t + 1) + B(t - 2)$

$t = 2 \Rightarrow 3 = 3A \Rightarrow A = 1$

$t = -1 \Rightarrow 3 = -3B \Rightarrow B = -1$

$\therefore \int \dfrac{3}{(t-2)(t+1)}\ dt$

$= \int (\dfrac{1}{t-2} - \dfrac{1}{t+1})\ dt$

$= \ln|t - 2| - \ln|t + 1| + c$

$= \ln\left|\dfrac{t-2}{t+1}\right| + c$

3 **a** $\dfrac{6x-11}{(2x+1)(x-3)} \equiv \dfrac{A}{2x+1} + \dfrac{B}{x-3}$

$6x - 11 \equiv A(x - 3) + B(2x + 1)$

$x = -\frac{1}{2} \Rightarrow -14 = -\frac{7}{2}A \Rightarrow A = 4$

$x = 3 \Rightarrow 7 = 7B \Rightarrow B = 1$

$\therefore \int \dfrac{6x-11}{(2x+1)(x-3)}\ dx$

$= \int (\dfrac{4}{2x+1} + \dfrac{1}{x-3})\ dx$

$= 2\ln|2x + 1| + \ln|x - 3| + c$

b $\dfrac{14-x}{x^2+2x-8} \equiv \dfrac{A}{x+4} + \dfrac{B}{x-2}$

$14 - x \equiv A(x - 2) + B(x + 4)$

$x = -4 \Rightarrow 18 = -6A \Rightarrow A = -3$

$x = 2 \Rightarrow 12 = 6B \Rightarrow B = 2$

$\therefore \int \dfrac{14-x}{x^2+2x-8}\ dx$

$= \int (\dfrac{2}{x-2} - \dfrac{3}{x+4})\ dx$

$= 2\ln|x - 2| - 3\ln|x + 4| + c$

c $\dfrac{6}{(2+x)(1-x)} \equiv \dfrac{A}{2+x} + \dfrac{B}{1-x}$

$6 \equiv A(1 - x) + B(2 + x)$

$x = -2 \Rightarrow 6 = 3A \Rightarrow A = 2$

$x = 1 \Rightarrow 6 = 3B \Rightarrow B = 2$

$\therefore \int \dfrac{6}{(2+x)(1-x)}\ dx$

$= \int (\dfrac{2}{2+x} + \dfrac{2}{1-x})\ dx$

$= 2\ln|2 + x| - 2\ln|1 - x| + c$

$= 2\ln\left|\dfrac{2+x}{1-x}\right| + c$

d $\dfrac{x+1}{5x^2-14x+8} \equiv \dfrac{A}{5x-4} + \dfrac{B}{x-2}$

$x + 1 \equiv A(x - 2) + B(5x - 4)$

$x = \frac{4}{5} \Rightarrow \frac{9}{5} = -\frac{6}{5}A \Rightarrow A = -\frac{3}{2}$

$x = 2 \Rightarrow 3 = 6B \Rightarrow B = \frac{1}{2}$

$\therefore \int \dfrac{x+1}{5x^2-14x+8}\ dx$

$= \int (\dfrac{\frac{1}{2}}{(x-2)} - \dfrac{\frac{3}{2}}{5x-4})\ dx$

$= \frac{1}{2}\ln|x - 2| - \frac{3}{10}\ln|5x - 4| + c$

4 **a** $x^2 - 6 \equiv A(x + 4)(x - 1) + B(x - 1) + C(x + 4)$

$x = -4 \Rightarrow 10 = -5B \Rightarrow B = -2$

$x = 1 \Rightarrow -5 = 5C \Rightarrow C = -1$

coeffs of $x^2 \Rightarrow A = 1$

b $= \int (1 - \dfrac{2}{x+4} - \dfrac{1}{x-1})\ dx$

$= x - 2\ln|x + 4| - \ln|x - 1| + c$

5 **a** $\dfrac{x^2 - x - 4}{(x+2)(x+1)^2} \equiv \dfrac{A}{x+2} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

$x^2 - x - 4 \equiv A(x+1)^2 + B(x+2)(x+1) + C(x+2)$

$x = -2 \quad \Rightarrow \quad A = 2$

$x = -1 \quad \Rightarrow \quad C = -2$

coeffs of $x^2 \quad \Rightarrow \quad 1 = A + B \quad \Rightarrow \quad B = -1$

$\therefore \dfrac{x^2 - x - 4}{(x+2)(x+1)^2} \equiv \dfrac{2}{x+2} - \dfrac{1}{x+1} - \dfrac{2}{(x+1)^2}$

b $= \int \left(\dfrac{2}{x+2} - \dfrac{1}{x+1} - \dfrac{2}{(x+1)^2}\right) \, dx$

$= 2 \ln|x+2| - \ln|x+1| + 2(x+1)^{-1} + c$

6 **a** $\dfrac{3x^2 - 5}{x^2 - 1} \equiv A + \dfrac{B}{x+1} + \dfrac{C}{x-1}$

$3x^2 - 5 \equiv A(x+1)(x-1) + B(x-1) + C(x+1)$

$x = -1 \quad \Rightarrow \quad -2 = -2B \quad \Rightarrow \quad B = 1$

$x = 1 \quad \Rightarrow \quad -2 = 2C \quad \Rightarrow \quad C = -1$

coeffs of $x^2 \quad \Rightarrow \quad A = 3$

$\therefore \int \dfrac{3x^2 - 5}{x^2 - 1} \, dx = \int \left(3 + \dfrac{1}{x+1} - \dfrac{1}{x-1}\right) \, dx$

$= 3x + \ln|x+1| - \ln|x-1| + c = 3x + \ln\left|\dfrac{x+1}{x-1}\right| + c$

b $\dfrac{x(4x+13)}{(2+x)^2(3-x)} \equiv \dfrac{A}{2+x} + \dfrac{B}{(2+x)^2} + \dfrac{C}{3-x}$

$x(4x+13) \equiv A(2+x)(3-x) + B(3-x) + C(2+x)^2$

$x = -2 \quad \Rightarrow \quad -10 = 5B \quad \Rightarrow \quad B = -2$

$x = 3 \quad \Rightarrow \quad 75 = 25C \quad \Rightarrow \quad C = 3$

coeffs of $x^2 \quad \Rightarrow \quad 4 = -A + C \quad \Rightarrow \quad A = -1$

$\therefore \int \dfrac{x(4x+13)}{(2+x)^2(3-x)} \, dx = \int \left(\dfrac{3}{3-x} - \dfrac{1}{2+x} - \dfrac{2}{(2+x)^2}\right) \, dx$

$= -3 \ln|3-x| - \ln|2+x| + 2(2+x)^{-1} + c$

c $\dfrac{x^2 - x + 1}{x^2 - 3x - 10} \equiv A + \dfrac{B}{x-5} + \dfrac{C}{x+2}$

$x^2 - x + 1 \equiv A(x-5)(x+2) + B(x+2) + C(x-5)$

$x = 5 \quad \Rightarrow \quad 21 = 7B \quad \Rightarrow \quad B = 3$

$x = -2 \quad \Rightarrow \quad 7 = -7C \quad \Rightarrow \quad C = -1$

coeffs of $x^2 \quad \Rightarrow \quad A = 1$

$\therefore \int \dfrac{x^2 - x + 1}{x^2 - 3x - 10} \, dx = \int \left(1 + \dfrac{3}{x-5} - \dfrac{1}{x+2}\right) \, dx$

$= x + 3 \ln|x-5| - \ln|x+2| + c$

d $\dfrac{2 - 6x + 5x^2}{x^2(1-2x)} \equiv \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{1-2x}$

$2 - 6x + 5x^2 \equiv Ax(1-2x) + B(1-2x) + Cx^2$

$x = \frac{1}{2} \quad \Rightarrow \quad \frac{1}{4} = \frac{1}{4}C \quad \Rightarrow \quad C = 1$

$x = 0 \quad \Rightarrow \quad B = 2$

coeffs of $x^2 \quad \Rightarrow \quad 5 = -2A + C \quad \Rightarrow \quad A = -2$

$\therefore \int \dfrac{2 - 6x + 5x^2}{x^2(1-2x)} \, dx = \int \left(\dfrac{1}{1-2x} - \dfrac{2}{x} + \dfrac{2}{x^2}\right) \, dx$

$= -\frac{1}{2} \ln|1-2x| - 2 \ln|x| - 2x^{-1} + c$

7 $\frac{3x-5}{(x-1)(x-2)} \equiv \frac{A}{x-1} + \frac{B}{x-2}$

$3x - 5 \equiv A(x - 2) + B(x - 1)$

$x = 1 \quad \Rightarrow \quad -2 = -A \quad \Rightarrow \quad A = 2$

$x = 2 \quad \Rightarrow \quad B = 1$

$\therefore \int_3^4 \frac{3x-5}{(x-1)(x-2)}\, dx = \int_3^4 \left(\frac{2}{x-1} + \frac{1}{x-2}\right) dx$

$= [2 \ln|x - 1| + \ln|x - 2|]_3^4$

$= (2 \ln 3 + \ln 2) - (2 \ln 2 + 0) = 2 \ln 3 - \ln 2$

8 **a** $\frac{x+3}{x(x+1)} \equiv \frac{A}{x} + \frac{B}{x+1}$

$x + 3 \equiv A(x + 1) + Bx$

$x = 0 \quad \Rightarrow \quad A = 3$

$x = -1 \quad \Rightarrow \quad 2 = -B \quad \Rightarrow \quad B = -2$

$\therefore \int_1^3 \frac{x+3}{x(x+1)}\, dx = \int_1^3 \left(\frac{3}{x} - \frac{2}{x+1}\right) dx$

$= [3 \ln|x| - 2 \ln|x + 1|]_1^3$

$= (3 \ln 3 - 2 \ln 4) - (0 - 2 \ln 2) = 3 \ln 3 - 2 \ln 2$

b $\frac{3x-2}{x^2+x-12} \equiv \frac{A}{x+4} + \frac{B}{x-3}$

$3x - 2 \equiv A(x - 3) + B(x + 4)$

$x = -4 \quad \Rightarrow \quad -14 = -7A \quad \Rightarrow \quad A = 2$

$x = 3 \quad \Rightarrow \quad 7 = 7B \quad \Rightarrow \quad B = 1$

$\therefore \int_0^2 \frac{3x-2}{x^2+x-12}\, dx = \int_0^2 \left(\frac{2}{x+4} + \frac{1}{x-3}\right) dx$

$= [2 \ln|x + 4| + \ln|x - 3|]_0^2$

$= (2 \ln 6 + 0) - (2 \ln 4 + \ln 3)$

$= 2(\ln 2 + \ln 3) - 4 \ln 2 - \ln 3 = \ln 3 - 2 \ln 2$

c $\frac{9}{2x^2-7x-4} \equiv \frac{A}{2x+1} + \frac{B}{x-4}$

$9 \equiv A(x - 4) + B(2x + 1)$

$x = -\frac{1}{2} \quad \Rightarrow \quad 9 = -\frac{9}{2}A \quad \Rightarrow \quad A = -2$

$x = 4 \quad \Rightarrow \quad 9 = 9B \quad \Rightarrow \quad B = 1$

$\therefore \int_1^2 \frac{9}{2x^2-7x-4}\, dx = \int_1^2 \left(\frac{1}{x-4} - \frac{2}{2x+1}\right) dx$

$= [\ln|x - 4| - \ln|2x + 1|]_1^2$

$= (\ln 2 - \ln 5) - (\ln 3 - \ln 3) = \ln 2 - \ln 5$

d $\frac{2x^2-7x+7}{x^2-2x-3} \equiv A + \frac{B}{x-3} + \frac{C}{x+1}$

$2x^2 - 7x + 7 \equiv A(x - 3)(x + 1) + B(x + 1) + C(x - 3)$

$x = 3 \quad \Rightarrow \quad 4 = 4B \quad \Rightarrow \quad B = 1$

$x = -1 \quad \Rightarrow \quad 16 = -4C \quad \Rightarrow \quad C = -4$

coeffs of $x^2 \quad \Rightarrow \quad A = 2$

$\therefore \int_0^2 \frac{2x^2-7x+7}{x^2-2x-3}\, dx = \int_0^2 \left(2 + \frac{1}{x-3} - \frac{4}{x+1}\right) dx$

$= [2x + \ln|x - 3| - 4 \ln|x + 1|]_0^2$

$= (4 + 0 - 4 \ln 3) - (0 + \ln 3 - 0) = 4 - 5 \ln 3$

e $\frac{5x+7}{(x+1)^2(x+3)} \equiv \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+3}$

$5x + 7 \equiv A(x + 1)(x + 3) + B(x + 3) + C(x + 1)^2$

$x = -1 \Rightarrow 2 = 2B \Rightarrow B = 1$

$x = -3 \Rightarrow -8 = 4C \Rightarrow C = -2$

coeffs of $x^2 \Rightarrow 0 = A + C \Rightarrow A = 2$

$\therefore \int_0^1 \frac{5x+7}{(x+1)^2(x+3)}\,dx = \int_0^1 \left(\frac{2}{x+1} + \frac{1}{(x+1)^2} - \frac{2}{x+3}\right) dx$

$= [2\ln|x+1| - (x+1)^{-1} - 2\ln|x+3|]_0^1$

$= (2\ln 2 - \frac{1}{2} - 2\ln 4) - (0 - 1 - 2\ln 3)$

$= \frac{1}{2} - 2\ln 2 + 2\ln 3$

f $\frac{2+x}{8-2x-x^2} \equiv \frac{A}{4+x} + \frac{B}{2-x}$

$2 + x \equiv A(2 - x) + B(4 + x)$

$x = -4 \Rightarrow -2 = 6A \Rightarrow A = -\frac{1}{3}$

$x = 2 \Rightarrow 4 = 6B \Rightarrow B = \frac{2}{3}$

$\therefore \int_{-1}^1 \frac{2+x}{8-2x-x^2}\,dx = \int_{-1}^1 \left(\frac{\frac{2}{3}}{2-x} - \frac{\frac{1}{3}}{4+x}\right) dx$

$= [-\frac{2}{3}\ln|2-x| - \frac{1}{3}\ln|4+x|]_{-1}^1$

$= (0 - \frac{1}{3}\ln 5) - (-\frac{2}{3}\ln 3 - \frac{1}{3}\ln 3)$

$= \ln 3 - \frac{1}{3}\ln 5$

9 **a** $\frac{1}{x^2-a^2} \equiv \frac{A}{x+a} + \frac{B}{x-a}$

$1 \equiv A(x - a) + B(x + a)$

$x = -a \Rightarrow 1 = -2aA \Rightarrow A = -\frac{1}{2a}$

$x = a \Rightarrow 1 = 2aB \Rightarrow B = \frac{1}{2a}$

$\therefore \frac{1}{x^2-a^2} \equiv \frac{1}{2a(x-a)} - \frac{1}{2a(x+a)}$

b $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\int \left(\frac{1}{x-a} - \frac{1}{x+a}\right) dx$

$= \frac{1}{2a}(\ln|x-a| - \ln|x+a|) + c$

$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$

c $\int \frac{1}{a^2-x^2}\,dx = -\int \frac{1}{x^2-a^2}\,dx$

$= -\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$

$= \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + c$

10 **a** $= [\frac{1}{6}\ln\left|\frac{x-3}{x+3}\right|]_{-1}^1$

$= \frac{1}{6}(\ln\frac{1}{2} - \ln 2)$

$= -\frac{1}{3}\ln 2$

b $= 4[\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|]_{-\frac{1}{2}}^{\frac{1}{2}}$

$= 2(\ln 3 - \ln\frac{1}{3})$

$= 4\ln 3$

c $= \frac{3}{2}[\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|]_0^1$

$= \frac{3}{8}(\ln\frac{1}{3} - 0)$

$= -\frac{3}{8}\ln 3$