C4 INTEGRATION — Answers - Worksheet E

1 **a** $u = x^2 + 1 \;\therefore\; \frac{du}{dx} = 2x$

$$\int 2x(x^2 - 1)^3 \, dx = \int u^3 \, du = \tfrac{1}{4}u^4 + c = \tfrac{1}{4}(x^2 + 1)^4 + c$$

b $u = \sin x \;\therefore\; \frac{du}{dx} = \cos x$

$$\int \sin^4 x \cos x \, dx = \int u^4 \, du = \tfrac{1}{5}u^5 + c = \tfrac{1}{5}\sin^5 x + c$$

c $u = 2 + x^3 \;\therefore\; \frac{du}{dx} = 3x^2$

$$\int 3x^2(2 + x^3)^2 \, dx = \int u^2 \, du = \tfrac{1}{3}u^3 + c = \tfrac{1}{3}(2 + x^3)^3 + c$$

d $u = x^2 \;\therefore\; \frac{du}{dx} = 2x$

$$\int 2x e^{x^2} \, dx = \int e^u \, du = e^u + c = e^{x^2} + c$$

e $u = x^2 + 3 \;\therefore\; \frac{du}{dx} = 2x$

$$\int \frac{x}{(x^2+3)^4} \, dx = \int \tfrac{1}{2}u^{-4} \, du = -\tfrac{1}{6}u^{-3} + c = -\frac{1}{6(x^2+3)^3} + c$$

f $u = \cos 2x \;\therefore\; \frac{du}{dx} = -2\sin 2x$

$$\int \sin 2x \cos^3 2x \, dx = \int -\tfrac{1}{2}u^3 \, du = -\tfrac{1}{8}u^4 + c = -\tfrac{1}{8}\cos^4 2x + c$$

g $u = x^2 - 2 \;\therefore\; \frac{du}{dx} = 2x$

$$\int \frac{3x}{x^2 - 2} \, dx = \int \frac{3}{2u} \, du = \tfrac{3}{2}\ln|u| + c = \tfrac{3}{2}\ln|x^2 - 2| + c$$

h $u = 1 - x^2 \;\therefore\; \frac{du}{dx} = -2x$

$$\int x\sqrt{1 - x^2} \, dx = \int -\tfrac{1}{2}u^{\frac{1}{2}} \, du = -\tfrac{1}{3}u^{\frac{3}{2}} + c = -\tfrac{1}{3}(1 - x^2)^{\frac{3}{2}} + c$$

i $u = \sec x \;\therefore\; \frac{du}{dx} = \sec x \tan x$

$$\int \sec^3 x \tan x \, dx = \int u^2 \, du = \tfrac{1}{3}u^3 + c = \tfrac{1}{3}\sec^3 x + c$$

j $u = x^2 + 2x \;\therefore\; \frac{du}{dx} = 2x + 2$

$$\int (x + 1)(x^2 + 2x)^3 \, dx = \int \tfrac{1}{2}u^3 \, du = \tfrac{1}{8}u^4 + c = \tfrac{1}{8}(x^2 + 2x)^4 + c$$

2 **a** **i** $u = 3$
ii $u = 4$

b $u = x^2 + 3 \;\therefore\; \frac{du}{dx} = 2x$

$$\int_0^1 2x(x^2 + 3)^2 \, dx = \int_3^4 u^2 \times \frac{du}{dx} \, dx = \int_3^4 u^2 \, du$$

c $$\int_0^1 2x(x^2 + 3)^2 \, dx = \int_3^4 u^2 \, du = [\tfrac{1}{3}u^3]_3^4 = \tfrac{64}{3} - 9 = 12\tfrac{1}{3}$$

3 **a** $u = x^2 - 3 \;\therefore\; \frac{du}{dx} = 2x$

$x = 1 \Rightarrow u = -2$

$x = 2 \Rightarrow u = 1$

$\int_1^2 x(x^2-3)^3 \, dx = \int_{-2}^{1} \tfrac{1}{2}u^3 \, du$

$= [\tfrac{1}{8}u^4]_{-2}^{1}$

$= \tfrac{1}{8}(1 - 16)$

$= -\tfrac{15}{8}$

b $u = \sin x \;\therefore\; \frac{du}{dx} = \cos x$

$x = 0 \Rightarrow u = 0$

$x = \tfrac{\pi}{6} \Rightarrow u = \tfrac{1}{2}$

$\int_0^{\frac{\pi}{6}} \sin^3 x \cos x \, dx = \int_0^{\frac{1}{2}} u^3 \, du$

$= [\tfrac{1}{4}u^4]_0^{\frac{1}{2}}$

$= \tfrac{1}{4}(\tfrac{1}{16} - 0)$

$= \tfrac{1}{64}$

c $u = x^2 + 1 \;\therefore\; \frac{du}{dx} = 2x$

$x = 0 \Rightarrow u = 1$

$x = 3 \Rightarrow u = 10$

$\int_0^3 \frac{4x}{x^2+1} \, dx = \int_1^{10} \frac{2}{u} \, du$

$= [2 \ln|u|]_1^{10}$

$= 2 \ln 10 - 0$

$= 2 \ln 10$

d $u = \tan x \;\therefore\; \frac{du}{dx} = \sec^2 x$

$x = -\tfrac{\pi}{4} \Rightarrow u = -1$

$x = \tfrac{\pi}{4} \Rightarrow u = 1$

$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x \sec^2 x \, dx = \int_{-1}^{1} u^2 \, du$

$= [\tfrac{1}{3}u^3]_{-1}^{1}$

$= \tfrac{1}{3}[1 - (-1)]$

$= \tfrac{2}{3}$

e $u = x^2 - 3 \;\therefore\; \frac{du}{dx} = 2x$

$x = 2 \Rightarrow u = 1$

$x = 3 \Rightarrow u = 6$

$\int_2^3 \frac{x}{\sqrt{x^2-3}} \, dx = \int_1^6 \tfrac{1}{2}u^{-\frac{1}{2}} \, du$

$= [u^{\frac{1}{2}}]_1^6$

$= \sqrt{6} - 1$

f $u = x^3 + 2 \;\therefore\; \frac{du}{dx} = 3x^2$

$x = -2 \Rightarrow u = -6$

$x = -1 \Rightarrow u = 1$

$\int_{-2}^{-1} x^2(x^3+2)^2 \, dx = \int_{-6}^{1} \tfrac{1}{3}u^2 \, du$

$= [\tfrac{1}{9}u^3]_{-6}^{1}$

$= \tfrac{1}{9}[1 - (-216)]$

$= 24\tfrac{1}{9}$

g $u = 1 + e^{2x} \;\therefore\; \frac{du}{dx} = 2e^{2x}$

$x = 0 \Rightarrow u = 2$

$x = 1 \Rightarrow u = 1 + e^2$

$\int_0^1 e^{2x}(1+e^{2x})^3 \, dx = \int_2^{1+e^2} \tfrac{1}{2}u^3 \, du$

$= [\tfrac{1}{8}u^4]_2^{1+e^2}$

$= \tfrac{1}{8}[(1+e^2)^4 - 16]$

$= \tfrac{1}{8}(1+e^2)^4 - 2$

h $u = x^2 - 4x \;\therefore\; \frac{du}{dx} = 2x - 4$

$x = 3 \Rightarrow u = -3$

$x = 5 \Rightarrow u = 5$

$\int_3^5 (x-2)(x^2-4x)^2 \, dx = \int_{-3}^{5} \tfrac{1}{2}u^2 \, du$

$= [\tfrac{1}{6}u^3]_{-3}^{5}$

$= \tfrac{1}{6}[125 - (-27)]$

$= 25\tfrac{1}{3}$

4 **a** $u = 4 - x^2 \quad \therefore \quad \frac{du}{dx} = -2x$

$x = 0 \Rightarrow u = 4$

$x = 2 \Rightarrow u = 0$

$$\int_0^2 x(4 - x^2)^3 \, dx = \int_4^0 u^3 \times (-\tfrac{1}{2} \tfrac{du}{dx}) \, du$$

$$= \int_0^4 \tfrac{1}{2}u^3 \, du$$

b $= [\tfrac{1}{8}u^4]_0^4$

$= \tfrac{1}{8}(256 - 0)$

$= 32$

5 **a** $u = 2 - x^2 \quad \therefore \quad \frac{du}{dx} = -2x$

$x = 0 \Rightarrow u = 2$

$x = 1 \Rightarrow u = 1$

$$\int_0^1 x e^{2-x^2} \, dx = \int_2^1 -\tfrac{1}{2} e^u \, du$$

$$= \int_1^2 \tfrac{1}{2} e^u \, du$$

$= [\tfrac{1}{2}e^u]_1^2$

$= \tfrac{1}{2}(e^2 - e)$

$= \tfrac{1}{2}e(e - 1)$

b $u = 1 + \cos x \quad \therefore \quad \frac{du}{dx} = -\sin x$

$x = 0 \Rightarrow u = 2$

$x = \frac{\pi}{2} \Rightarrow u = 1$

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos x} \, dx = \int_2^1 -\frac{1}{u} \, du$$

$$= \int_1^2 \frac{1}{u} \, du$$

$= [\ln|u|]_1^2$

$= \ln 2 - 0$

$= \ln 2$

6 **a** $u = \sin x \quad \therefore \quad \frac{du}{dx} = \cos x$

$$\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx$$

$$= \int \frac{1}{u} \times \frac{du}{dx} \, dx$$

$$= \int \frac{1}{u} \, du$$

$= \ln|u| + c$

$= \ln|\sin x| + c$

b $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$

$u = \cos x \quad \therefore \quad \frac{du}{dx} = -\sin x$

$$\int \frac{\sin x}{\cos x} \, dx = \int \frac{1}{u} \times (-\frac{du}{dx}) \, dx$$

$$= \int -\frac{1}{u} \, du$$

$= -\ln|u| + c$

$= -\ln|\cos x| + c$

$= \ln(|\cos x|)^{-1} + c$

$= \ln|\sec x| + c$

c $= [\tfrac{1}{2}\ln|\sec 2x|]_0^{\frac{\pi}{6}}$

$= \tfrac{1}{2}(\ln 2 - 0)$

$= \tfrac{1}{2}\ln 2$

7

a $= \frac{1}{4}(x^3 - 2)^4 + c$

b $= e^{\sin x} + c$

c $= \frac{1}{2}\int \frac{2x}{x^2+1}\, dx$

$= \frac{1}{2}\ln|x^2 + 1| + c$

$[= \frac{1}{2}\ln(x^2 + 1) + c\ ]$

d $= \frac{1}{3}(x^2 + 3x)^3 + c$

e $= \frac{1}{2}\int 2x(x^2+4)^{\frac{1}{2}}\, dx$

$= \frac{1}{2} \times \frac{2}{3}(x^2+4)^{\frac{3}{2}} + c$

$= \frac{1}{3}(x^2+4)^{\frac{3}{2}} + c$

f $= -\int \cot^3 x\,(-\operatorname{cosec}^2 x)\, dx$

$= -\frac{1}{4}\cot^4 x + c$

g $= \ln|1 + e^x| + c$

$[= \ln(1 + e^x) + c\ ]$

h $= \frac{1}{2}\int \frac{2\cos 2x}{3+\sin 2x}\, dx$

$= \frac{1}{2}\ln|3 + \sin 2x| + c$

i $= \frac{1}{4}\int \frac{4x^3}{(x^4-2)^2} + c$

$= \frac{1}{4} \times [-(x^4 - 2)^{-1}] + c$

$= -\frac{1}{4(x^4-2)} + c$

j $= \frac{1}{4}(\ln x)^4 + c$

k $= \frac{2}{3}\int \frac{3}{2}x^{\frac{1}{2}}(1+x^{\frac{3}{2}})^2\, dx$

$= \frac{2}{3} \times \frac{1}{3}(1+x^{\frac{3}{2}})^3 + c$

$= \frac{2}{9}(1+x^{\frac{3}{2}})^3 + c$

l $= -\frac{1}{2}\int -2x(5-x^2)^{-\frac{1}{2}}\, dx$

$= -\frac{1}{2} \times 2(5-x^2)^{\frac{1}{2}} + c$

$= -\sqrt{5-x^2} + c$

8

a $= -\int_0^{\frac{\pi}{2}} (-\sin x)(1 + \cos x)^2\, dx$

$= -[\frac{1}{3}(1 + \cos x)^3]_0^{\frac{\pi}{2}}$

$= -\frac{1}{3}(1 - 8)$

$= \frac{7}{3}$

b $= -\frac{1}{2}\int_{-1}^{0} \frac{-2e^{2x}}{2-e^{2x}}\, dx$

$= -\frac{1}{2}[\ln|2 - e^{2x}|]_{-1}^{0}$

$= -\frac{1}{2}[0 - \ln(2 - e^{-2})]$

$= \frac{1}{2}\ln(2 - e^{-2})$

c $= -\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (-\cot x \operatorname{cosec} x)\operatorname{cosec}^3 x\, dx$

$= -[\frac{1}{4}\operatorname{cosec}^4 x]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$

$= -\frac{1}{4}(4 - 16)$

$= 3$

d $= \frac{1}{2}\int_2^4 \frac{2x+2}{x^2+2x+8}\, dx$

$= \frac{1}{2}[\ln|x^2 + 2x + 8|]_2^4$

$= \frac{1}{2}(\ln 32 - \ln 16)$

$= \frac{1}{2}\ln 2$

9

$u = x + 1 \quad \therefore x = u - 1, \quad \frac{du}{dx} = 1$

$\int x(x + 1)^3\, dx = \int (u - 1)u^3\, du$

$= \int (u^4 - u^3)\, du$

$= \frac{1}{5}u^5 - \frac{1}{4}u^4 + c$

$= \frac{1}{5}(x + 1)^5 - \frac{1}{4}(x + 1)^4 + c$

$= \frac{1}{20}(x + 1)^4[4(x + 1) - 5] + c$

$= \frac{1}{20}(4x - 1)(x + 1)^4 + c$

10 **a** $u = 2x - 1 \quad \therefore x = \frac{1}{2}(u + 1), \quad \frac{du}{dx} = 2$

$$\int x(2x-1)^4 \, dx = \int \tfrac{1}{2}(u+1)u^4 \times \tfrac{1}{2} \, du$$
$$= \tfrac{1}{4}\int (u^5 + u^4) \, du$$
$$= \tfrac{1}{4}(\tfrac{1}{6}u^6 + \tfrac{1}{5}u^5) + c$$
$$= \tfrac{1}{4}[\tfrac{1}{6}(2x-1)^6 + \tfrac{1}{5}(2x-1)^5] + c$$
$$= \tfrac{1}{120}(2x-1)^5[5(2x-1)+6] + c$$
$$= \tfrac{1}{120}(10x+1)(2x-1)^5 + c$$

b $u^2 = 1 - x \quad \therefore x = 1 - u^2, \quad \frac{dx}{du} = -2u$

$$\int x\sqrt{1-x} \, dx = \int (1-u^2)u \times (-2u) \, du$$
$$= 2\int (u^4 - u^2) \, du$$
$$= 2(\tfrac{1}{5}u^5 - \tfrac{1}{3}u^3) + c$$
$$= 2[\tfrac{1}{5}(1-x)^{\frac{5}{2}} - \tfrac{1}{3}(1-x)^{\frac{3}{2}} + c$$
$$= \tfrac{2}{15}(1-x)^{\frac{3}{2}}[3(1-x) - 5] + c$$
$$= -\tfrac{2}{15}(2+3x)(1-x)^{\frac{3}{2}} + c$$

c $x = \sin u \quad \therefore \frac{dx}{du} = \cos u$

$$\int \frac{1}{(1-x^2)^{\frac{3}{2}}} \, dx = \int \frac{1}{\cos^3 u} \times \cos u \, du$$
$$= \int \sec^2 u \, du$$
$$= \tan u + c$$
$$= \frac{\sin u}{\cos u} + c$$
$$= \frac{x}{\sqrt{1-x^2}} + c$$

d $x = u^2 \quad \therefore \frac{dx}{du} = 2u$

$$\int \frac{1}{\sqrt{x}-1} \, dx = \int \frac{1}{u-1} \times 2u \, du$$
$$= \int \frac{2(u-1)+2}{u-1} \, du$$
$$= \int (2 + \frac{2}{u-1}) \, du$$
$$= 2u + 2\ln|u-1| + c$$
$$= 2\sqrt{x} + 2\ln|\sqrt{x} - 1| + c$$

e $u = 2x + 3 \quad \therefore x = \frac{1}{2}u - \frac{3}{2}, \quad \frac{du}{dx} = 2$

$$\int (x+1)(2x+3)^3 \, dx$$
$$= \int (\tfrac{1}{2}u - \tfrac{1}{2})u^3 \times \tfrac{1}{2} \, du$$
$$= \tfrac{1}{4}\int (u^4 - u^3) \, du$$
$$= \tfrac{1}{4}(\tfrac{1}{5}u^5 - \tfrac{1}{4}u^4) + c$$
$$= \tfrac{1}{4}[\tfrac{1}{5}(2x+3)^5 - \tfrac{1}{4}(2x+3)^4] + c$$
$$= \tfrac{1}{80}(2x+3)^4[4(2x+3) - 5] + c$$
$$= \tfrac{1}{80}(8x+7)(2x+3)^4 + c$$

f $u^2 = x - 2 \quad \therefore x = u^2 + 2, \quad \frac{dx}{du} = 2u$

$$\int \frac{x^2}{\sqrt{x-2}} \, dx = \int \frac{(u^2+2)^2}{u} \times 2u \, du$$
$$= 2\int (u^4 + 4u^2 + 4) \, du$$
$$= 2(\tfrac{1}{5}u^5 + \tfrac{4}{3}u^3 + 4u) + c$$
$$= 2[\tfrac{1}{5}(x-2)^{\frac{5}{2}} + \tfrac{4}{3}(x-2)^{\frac{3}{2}} + 4(x-2)^{\frac{1}{2}}] + c$$
$$= \tfrac{2}{15}(x-2)^{\frac{1}{2}}[3(x-2)^2 + 20(x-2) + 60] + c$$
$$= \tfrac{2}{15}(3x^2 + 8x + 32)(x-2)^{\frac{1}{2}} + c$$

11 **a** $x = \sin u \ \therefore \ \frac{dx}{du} = \cos u$

$x = 0 \Rightarrow u = 0$

$x = \frac{1}{2} \Rightarrow u = \frac{\pi}{6}$

$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\ dx = \int_0^{\frac{\pi}{6}} \frac{1}{\cos u} \times \cos u\ du$

$= \int_0^{\frac{\pi}{6}} du$

$= [u]_0^{\frac{\pi}{6}}$

$= \frac{\pi}{6} - 0$

$= \frac{\pi}{6}$

b $u = 2 - x \ \therefore \ x = 2 - u, \ \frac{du}{dx} = -1$

$x = 0 \Rightarrow u = 2$

$x = 2 \Rightarrow u = 0$

$\int_0^2 x(2-x)^3\ dx = \int_2^0 (2-u)u^3 \times (-1)\ du$

$= \int_0^2 (2u^3 - u^4)\ du$

$= [\frac{1}{2}u^4 - \frac{1}{5}u^5]_0^2$

$= (8 - \frac{32}{5}) - (0)$

$= \frac{8}{5}$

c $x = 2\sin u \ \therefore \ \frac{dx}{du} = 2\cos u$

$x = 0 \Rightarrow u = 0$

$x = 1 \Rightarrow u = \frac{\pi}{6}$

$\int_0^1 \sqrt{4-x^2}\ dx$

$= \int_0^{\frac{\pi}{6}} 2\cos u \times 2\cos u\ du$

$= \int_0^{\frac{\pi}{6}} 4\cos^2 u\ du$

$= \int_0^{\frac{\pi}{6}} (2 + 2\cos 2u)\ du$

$= [2u + \sin 2u]_0^{\frac{\pi}{6}}$

$= (\frac{\pi}{3} + \frac{\sqrt{3}}{2}) - (0)$

$= \frac{1}{6}(2\pi + 3\sqrt{3})$

d $x = 3\tan u \ \therefore \ \frac{dx}{du} = 3\sec^2 u$

$x = 0 \Rightarrow u = 0$

$x = 3 \Rightarrow u = \frac{\pi}{4}$

$\int_0^3 \frac{x^2}{x^2+9}\ dx = \int_0^{\frac{\pi}{4}} \frac{9\tan^2 u}{9\sec^2 u} \times 3\sec^2 u\ du$

$= 3\int_0^{\frac{\pi}{4}} \tan^2 u\ du$

$= 3\int_0^{\frac{\pi}{4}} (\sec^2 u - 1)\ du$

$= 3[\tan u - u]_0^{\frac{\pi}{4}}$

$= 3[(1 - \frac{\pi}{4}) - (0)]$

$= \frac{3}{4}(4 - \pi)$