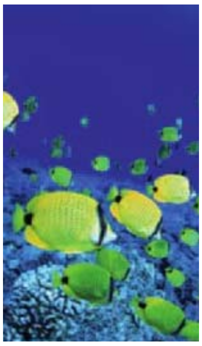





Review Exercise

1 Express $\frac{2x - 1}{(x - 1)(2x - 3)}$ in partial fractions. **E**

2 It is given that $f(x) = \frac{3x + 7}{(x + 1)(x + 2)(x + 3)}$.

Express f(x) as the sum of three partial fractions. **E**

3 Given that $f(x) = \frac{2}{(2 - x)(1 + x)^2}$,

express f(x) in the form

$\frac{A}{(2 - x)} + \frac{B}{(1 + x)} + \frac{C}{(1 + x)^2}$. **E**

4 $\frac{14x^2 + 13x + 2}{(x + 1)(2x + 1)^2} \equiv \frac{A}{x + 1} + \frac{B}{2x + 1} + \frac{C}{(2x + 1)^2}$.

Find the values of the constants A, B and C. **E**

5 $f(x) = \frac{x^2 + 6x + 7}{(x + 2)(x + 3)}, x \in \mathbb{R}$.

Given that $f(x) = A + \frac{B}{(x + 2)} + \frac{C}{(x + 3)}$

find the values of A, B and C. **E**

6 Given that $f(x) = \frac{11 - 5x^2}{(x + 1)(2 - x)}$, find constants A and B such that

$f(x) = 5 + \frac{A}{(x + 1)} + \frac{B}{(2 - x)}$. **E**

7 $f(x) = \frac{9 - 3x - 12x^2}{(1 - x)(1 + 2x)}$.

Given that $f(x) = A + \frac{B}{(1 - x)} + \frac{C}{(1 + 2x)}$,

find the values of the constants A, B and C. **E**

8 Use the Binomial theorem to expand $\sqrt{(4 - 9x)}$, $|x| < \frac{4}{9}$, in ascending powers of x, as far as the term in x^3, simplifying each term. **E**

9 $f(x) = (2 - 5x)^{-2}$, $|x| < \frac{2}{5}$.

Find the binomial expansion of f(x), in ascending powers of x, as far as the term in x^3, giving each coefficient as a simplified fraction. **E**

10 $f(x) = (3 + 2x)^{-3}$, $|x| < \frac{3}{2}$.

Find the binomial expansion of f(x), in ascending powers of x, as far as the term in x^3. Give each coefficient as a simplified fraction. **E**

11 $f(x) = \frac{1}{\sqrt{(1-x)}} - \sqrt{(1+x)}, \ -1 < x < 1$

Find the series expansion of f(x) in ascending powers of x up to and including the term in x^3. *E*

12 Given that

$$\frac{3+5x}{(1+3x)(1-x)} = \frac{A}{(1+3x)} + \frac{B}{(1-x)},$$

a find the values of the constants A and B.

b Hence or otherwise find the series expansion, in ascending powers of x, up to and including the term in x^2, of $\frac{3+5x}{(1+3x)(1-x)}$.

c State, with a reason, whether your series expansion in part **b** is valid for $x = \frac{1}{2}$. *E*

13 $f(x) = \frac{3x-1}{(1-2x)^2}, \ |x| < \frac{1}{2}$.

Given that, for $x \neq \frac{1}{2}$,

$$\frac{3x-1}{(1-2x)^2} = \frac{A}{(1-2x)} + \frac{B}{(1-2x)^2},$$

where A and B are constants,

a find the values of A and B.

b Hence or otherwise find the series expansion of f(x), in ascending powers of x, up to and including the term in x^3, simplifying each term. *E*

14 $f(x) = \frac{3x^2+16}{(1-3x)(2+x)^2}$

$$= \frac{A}{(1-3x)} + \frac{B}{(2+x)} + \frac{C}{(2+x)^2}, \ |x| < \frac{1}{3}.$$

a Find the values of A and C and show that $B = 0$.

b Hence, or otherwise, find the series expansion of f(x), in ascending powers of x, up to and including the term in x^3. Simplify each term. *E*

15 $f(x) = \frac{25}{(3+2x)^2(1-x)}$.

a Express f(x) as a sum of partial fractions.

b Find the series expansion of f(x) in ascending powers of x up to and including the term in x^2. Give each coefficient as a simplified fraction. *E*

16 When $(1 + ax)^n$ is expanded as a series in ascending powers of x, the coefficients of x and x^2 are -6 and 45 respectively.

a Find the value of a and the value of n.

b Find the coefficient of x^3.

c Find the set of values of x for which the expansion is valid.

E **[adapted]**

17 **a** Find the binomial expansion of $\sqrt{(1-x)}$, in ascending powers of x up to and including the term in x^3.

b By substituting a suitable value for x in this expansion, find an approximation to $\sqrt{0.9}$, giving your answer to 6 decimal places.

18 In the binomial expansion, in ascending powers of x, of $(1 + ax)^n$, where a and n are constants, the coefficient of x is 15. The coefficients of x^2 and of x^3 are equal.

a Find the value of a and the value of n.

b Find the coefficient of x^3.

19 The vectors **u** and **v** are given by

$$\mathbf{u} = 5\mathbf{i} - 4\mathbf{j} + s\mathbf{k}, \ \mathbf{v} = 2\mathbf{i} + t\mathbf{j} - 3\mathbf{k}$$

a Given that the vectors **u** and **v** are perpendicular, find a relation between the scalars s and t.

b Given instead that the vectors **u** and **v** are parallel, find the values of the scalars s and t. *E*

20 Find the angle between the vectors **a** and **b** given that $|\mathbf{a}| = 4$, $|\mathbf{b}| = 4$ and $|\mathbf{a} - \mathbf{b}| = 7$.

E **[adapted]**

21 The position vectors of the points A and B relative to an origin O are $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $5\mathbf{i} + 6\mathbf{j} + 5\mathbf{k}$ respectively. Find the position vector of the point P which lies on AB produced such that $AP = 3BP$.

E **[adapted]**

22 The points A and B have coordinates $(2t, 10, 1)$ and $(3t, 2t, 5)$ respectively.

a Find $|\overrightarrow{AB}|$.

b By differentiating $|\overrightarrow{AB}|^2$, find the value of t for which $|\overrightarrow{AB}|$ is a minimum.

c Find the minimum value of $|\overrightarrow{AB}|$.

23 The line l_1 has vector equation $\mathbf{r} = 11\mathbf{i} + 5\mathbf{j} + 6\mathbf{k} + \lambda(4\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})$ and the line l_2 has vector equation $\mathbf{r} = 24\mathbf{i} + 4\mathbf{j} + 13\mathbf{k} + \mu(7\mathbf{i} + \mathbf{j} + 5\mathbf{k})$, where λ and μ are parameters.

a Show that the lines l_1 and l_2 intersect.

b Find the coordinates of their point of intersection.

Given that θ is the acute angle between l_1 and l_2

c find the value of $\cos\theta$. Give your answer in the form $k\sqrt{3}$, where k is a simplified fraction. *E*

24 The line l_1 has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

and the line l_2 has equation

$$\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

a Show that l_1 and l_2 do not meet.

A is the point on l_1 where $\lambda = 1$ and B is the point on l_2 where $\mu = 2$.

b Find the cosine of the acute angle between AB and l_1. *E*

25 The line l_1 has vector equation

$$\mathbf{r} = 8\mathbf{i} + 12\mathbf{j} + 14\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} - \mathbf{k}).$$

The points A, with coordinates $(4, 8, a)$, and B, with coordinates $(b, 13, 13)$, lie on this line.

a Find the values of a and b.

Given that the point O is the origin, and that the point P lies on l_1 such that OP is perpendicular to l_1,

b find the coordinates of P.

c Hence find the distance OP, giving your answer as a simplified surd. *E*

26 The line l_1 has equation $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$

and the line l_2 has equation

$$\mathbf{r} = \begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Find, by calculation,

a the coordinates of B, the point of intersection of l_1 and l_2,

b the value of $\cos\theta$, where θ is the acute angle between l_1 and l_2. (Give your answer as a simplified fraction.)

The point A, which lies on l_1 has position vector $\mathbf{a} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. The point C, which lies on l_2, has position vector $\mathbf{c} = 5\mathbf{i} - \mathbf{j} - 2\mathbf{k}$. The point D lies in the plane ABC and $ABCD$ is a parallelogram .

c Show that $|\overrightarrow{AB}| = |\overrightarrow{BC}|$.

d Find the position vector of the point D. *E*

27 The points A and B have position vectors $5\mathbf{j} + 11\mathbf{k}$ and $c\mathbf{i} + d\mathbf{j} + 21\mathbf{k}$ respectively, where c and d are constants.

The line AB has vector equation $\mathbf{r} = 5\mathbf{j} + 11\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} + 5\mathbf{k})$.

a Find the value of c and the value of d.

The point P lies on the line AB, and $\overrightarrow{OP}$ is perpendicular to the line AB, where O is the origin.

b Find the position vector of P.

c Find the area of triangle OAB, giving your answer to 3 significant figures. *E*

28 The points A and B have position vectors $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ respectively.

a Find $|\overrightarrow{AB}|$.

b Find a vector equation for the line l_1 which passes through the points A and B.

A second line l_2 has vector equation $\mathbf{r} = 6\mathbf{i} + 4\mathbf{j} - 3\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} - \mathbf{k})$.

c Show that the line l_2 also passes through B.

d Find the size of the acute angle between l_1 and l_2.

e Hence, or otherwise, find the shortest distance from A to l_2. *E*

29 The point A, with coordinates $(0, a, b)$ lies on the line l_1, which has equation $\mathbf{r} = 6\mathbf{i} + 19\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$.

a Find the values of a and b.

The point P lies on l_1 and is such that OP is perpendicular to l_1, where O is the origin.

b Find the position vector of point P.

Given that B has coordinates $(5, 15, 1)$,

c show that the points A, P and B are collinear and find the ratio $AP : PB$. *E*

30 The point A has position vector $\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and the point B has position vector $\mathbf{b} = \mathbf{i} + \mathbf{j} - 4\mathbf{k}$, relative to an origin O.

a Find the position vector of the point C, with position vector $\mathbf{c}$, given by $\mathbf{c} = \mathbf{a} + \mathbf{b}$.

b Show that $OACB$ is a rectangle, and find its exact area.

The diagonals of the rectangle, AB and OC meet at the point D.

c Write down the position vector of the point D.

d Find the size of the angle ADC. *E*

31 Relative to a fixed origin O, the point A has position vector $5\mathbf{j} + 5\mathbf{k}$ and the point B has position vector $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

a Find a vector equation of the line L which passes through A and B.

The point C lies on the line L and OC is perpendicular to L.

b Find the position vector of C.

The points O, B and A together with the point D lie at the vertices of parallelogram $OBAD$.

c Find the position vector of D.

d Find the area of the parallelogram $OBAD$. *E*

32 Find the gradient of the curve

$$3x^3 - 2x^2y + y^3 = 17$$

at the point with coordinates $(2, 1)$. *E*

33 A curve has equation

$$x^2 + 2xy - 3y^2 + 16 = 0.$$

Find the coordinates of the points on the curve where $\frac{dy}{dx} = 0$. *E*

34 A curve C is described by the equation

$$3x^2 - 2y^2 + 2x - 3y + 5 = 0.$$

Find an equation of the normal to C at the point $(0, 1)$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers. *E*

35 A curve C is described by the equation

$$3x^2 + 4y^2 - 2x + 6xy - 5 = 0.$$

Find an equation of the tangent to C at the point $(1, -2)$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers. *E*

36 A set of curves is given by the equation $\sin x + \cos y = 0.5$.

a Use implicit differentiation to find an expression for $\frac{dy}{dx}$.

For $-\pi < x < \pi$ and $-\pi < y < \pi$

b find the coordinates of the points where $\frac{dy}{dx} = 0$. *E*

37 **a** Given that $y = 2^x$, and using the result $2^x = e^{x \ln 2}$, or otherwise, show that $\frac{dy}{dx} = 2^x \ln 2$.

b Find the gradient of the curve with equation $y = 2^{x^2}$ at the point with coordinates (2, 16). *E*

38 Find the coordinates of the minimum point on the curve with equation $y = x2^x$. *E*

39 The value £V of a car t years after the 1st January 2001 is given by the formula $V = 10000 \times (1.5)^{-t}$.

a Find the value of the car on 1st January 2005

b Find the value of $\frac{dV}{dt}$ when $t = 4$. *E*

40 A spherical balloon is being inflated in such a way that the rate of increase of its volume, V cm^3, with respect to time t seconds is given by

$\frac{dV}{dt} = \frac{k}{V}$, where k is a positive constant.

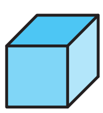
Given that the radius of the balloon is r cm, and that $V = \frac{4}{3}\pi r^3$,

a prove that r satisfies the differential equation

$\frac{dr}{dt} = \frac{B}{r^5}$, where B is a constant.

b Find a general solution of the differential equation obtained in part **a**. *E*

41



At time t seconds the length of the side of a cube is x cm, the surface area of the cube is S cm^2, and the volume of the cube is V cm^3.

The surface area of the cube is increasing at a constant rate of 8 cm^2 s^{-1}.

Show that

a $\frac{dx}{dt} = \frac{k}{x}$, where k is a constant to be found,

b $\frac{dV}{dt} = 2V^{\frac{1}{3}}$.

Given that $V = 8$ when $t = 0$,

c solve the differential equation in part **b**, and find the value of t when $V = 16\sqrt{2}$. *E*

42 Liquid is poured into a container at a constant rate of 30 cm^3 s^{-1}. At time t seconds liquid is leaking from the container at a rate of $\frac{2}{15}V$ cm^3 s^{-1}, where V cm^3 is the volume of liquid in the container at that time.

a Show that

$$-15\frac{dV}{dt} = 2V - 450.$$

Given that $V = 1000$ when $t = 0$,

b find the solution of the differential equation, in the form $V = f(t)$.

c Find the limiting value of V as $t \to \infty$. *E*

43 Liquid is pouring into a container at a constant rate of 20 cm^3 s^{-1} and is leaking out at a rate proportional to the volume of the liquid already in the container.

a Explain why, at time t seconds, the volume, V cm^3, of liquid in the container satisfies the differential equation

$$\frac{dV}{dt} = 20 - kV,$$

where k is a positive constant.

The container is initial empty.

b By solving the differential equation, show that

$$V = A + Be^{-kt},$$

giving the values of A and B in terms of k.

Given also that $\frac{dV}{dt} = 10$ when $t = 5$,

c find the volume of liquid in the container at 10 s after the start. *E*

44 **a** Express $\frac{2x - 1}{(x - 1)(2x - 3)}$ in partial fractions.

b Given that $x \geqslant 2$, find the general solution of the differential equation

$$(2x - 3)(x - 1)\frac{dy}{dx} = (2x - 1)y.$$

c Hence find the particular solution of this differential equation that satisfies $y = 10$ at $x = 2$, giving your answer in the form $y = f(x)$. *E*

45 The rate of decrease of the concentration of a drug in the blood stream is proportional to the concentration C of that drug which is present at that time. The time t is measured in hours from the administration of the drug and C is measured in micrograms per litre.

a Show that this process is described by the differential equation $\frac{dC}{dt} = -kC$, explaining why k is a positive constant.

b Find the general solution of the differential equation, in the form $C = f(t)$.

After 4 hours, the concentration of the drug in the bloodstream is reduced to 10% of its starting value C_0.

c Find the exact value of k. *E*

46 A radioactive isotope decays in such a way that the rate of change of the number, N, of radioactive atoms present after t days, is proportional to N.

a Write down a differential equation relating N and t.

b Show that the general solution may be written as $N = Ae^{-kt}$, where A and k are positive constants.

Initially the number of radioactive atoms present is 7×10^{18} and 8 days later the number present is 3×10^{17}.

c Find the value of k.

d Find the number of radioactive atoms present after a further 8 days. *E*

47 The volume of a spherical balloon of radius r cm is V cm^3, where $V = \frac{4}{3}\pi r^3$.

a Find $\frac{dV}{dr}$.

The volume of the balloon increases with time t seconds according to the formula

$$\frac{dV}{dt} = \frac{1000}{(2t + 1)^2}, \qquad t \geqslant 0.$$

b Using the chain rule, or otherwise, find an expression in terms of r and t for $\frac{dr}{dt}$.

c Given that $V = 0$ when $t = 0$, solve the differential equation $\frac{dV}{dt} = \frac{1000}{(2t + 1)^2}$, to obtain V in terms of t.

d Hence, at time $t = 5$,

i find the radius of the balloon, giving your answer to 3 significant figures,

ii show that the rate of increase of the radius of the balloon is approximately 2.90×10^{-2} cm s^{-1}. *E*

48 A population growth is modelled by the differential equation $\frac{dP}{dt} = kP$, where P is the population, t is the time measured in days and k is a positive constant.

Given that the initial population is P_0,

a solve the differential equation, giving P in terms of P_0, k and t.

Given also that $k = 2.5$,

b find the time taken, to the nearest minute, for the population to reach $2P_0$.

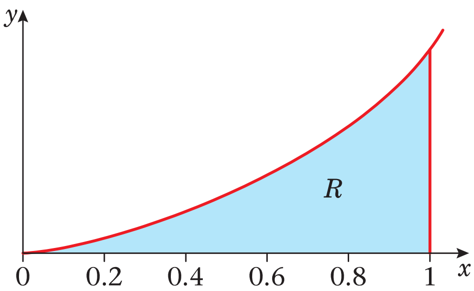
In an improved model the differential equation is given as $\frac{dP}{dt} = \lambda P \cos \lambda t$, where P is the population, t is the time measured in days and λ is a positive constant.

Given, again, that the initial population is P_0 and that time is measured in days,

c solve the second differential equation, giving P in terms of P_0, λ and t. Given also that $\lambda = 2.5$,

d find the time taken, to the nearest minute, for the population to reach $2P_0$ for the first time, using the improved model. *E*

49



The diagram shows the graph of the curve with equation

$$y = xe^{2x}, x \geqslant 0.$$

The finite region R bounded by the lines $x = 1$, the x-axis and the curve is shown shaded in the diagram.

a Use integration to find the exact value of the area for R.

b Complete the table with the values of y corresponding to $x = 0.4$ and 0.8.

x	0	0.2	0.4	0.6	0.8	1
$y = xe^{2x}$	0	0.29836		1.99207		7.38906

c Use the trapezium rule with all the values in the table to find an approximate value for this area, giving your answer to 4 significant figures. *E*

50 **a** Given that $y = \sec x$, complete the table with the values of y corresponding to $x = \frac{\pi}{16}, \frac{\pi}{8}$ and $\frac{\pi}{4}$.

x	0	$\frac{\pi}{16}$	$\frac{\pi}{8}$	$\frac{3\pi}{16}$	$\frac{\pi}{4}$
y	1			1.20269	

b Use the trapezium rule, with all the values for y in the completed table, to obtain an estimate for $\int_0^{\frac{\pi}{4}} \sec x \, dx$.

Show all the steps of your working and give your answer to 4 decimal places.

The exact value of $\int_0^{\frac{\pi}{4}} \sec x \, dx$ is $\ln(1 + \sqrt{2})$.

c Calculate the % error in using the estimate you obtained in part **b**. *E*

51 $I = \int_0^5 e^{\sqrt{(3x+1)}} \, dx$.

a Given that $y = e^{\sqrt{(3x+1)}}$, complete the table with the values of y corresponding to $x = 2$, 3 and 4.

x	0	1	2	3	4	5
y	e^1	e^2				e^4

b Use the trapezium rule, with all the values of y in the completed table, to obtain an estimate for the original integral I, giving your answer to 4 significant figures.

c Use the substitution $t = \sqrt{(3x+1)}$ to show that I may be expressed as $\int_a^b kte^t \, dt$, giving the values of a, b and k.

d Use integration by parts to evaluate this integral, and hence find the value of I correct to 4 significant figures, showing all the steps in your working. *E*

52 The following is a table of values for $y = \sqrt{(1 + \sin x)}$, where x is in radians.

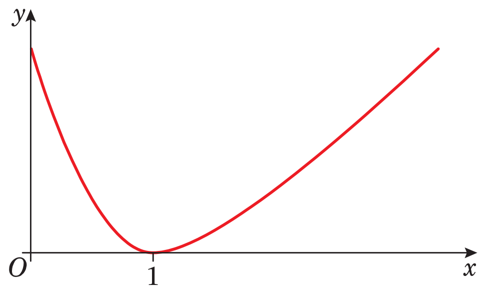
x	0	0.5	1	1.5	2
y	1	1.216	p	1.413	q

a Find the value of p and the value of q.

b Use the trapezium rule and all the values of y in the completed table to obtain an estimate of I, where

$$I = \int_0^2 \sqrt{(1 + \sin x)} \, dx$$ *E*

53



The figure shows a sketch of the curve with equation $y = (x - 1)\ln x, x > 0$.

a Copy and complete the table with the values of y corresponding to $x = 1.5$ and $x = 2.5$.

x	1	1.5	2	2.5	3
y	0		ln 2		2 ln 3

Given that $I = \int_1^3 (x - 1)\ln x \, dx$,

b use the trapezium rule

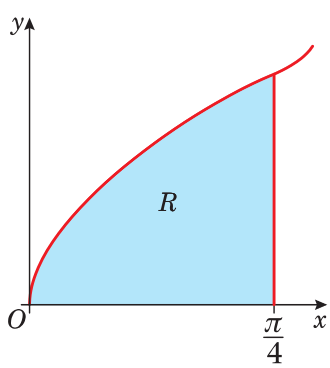
i with values of y at $x = 1$, 2 and 3 to find an approximate value for I to 4 significant figures,

ii with values of y at $x = 1$, 1.5, 2, 2.5 and 3 to find another approximate value for I to 4 significant figures.

c Explain, with reference to the figure, why an increase in the number of values improves the accuracy of the approximation.

d Show, by integration, that the exact value of $\int_1^3 (x - 1)\ln x \, dx$ is $\frac{3}{2}\ln 3$. *E*

54



The figure shows part of the curve with equation $\sqrt{(\tan x)}$. The finite region R, which is bounded by the curve, the x-axis and the line $x = \frac{\pi}{4}$, is shown shaded in the figure.

a Given that $y = \sqrt{(\tan x)}$, copy and complete the table with the values of y corresponding to $x = \frac{\pi}{16}, \frac{\pi}{8}$ and $\frac{3\pi}{16}$, giving your answers to 5 decimal places.

x	0	$\frac{\pi}{16}$	$\frac{\pi}{8}$	$\frac{3\pi}{16}$	$\frac{\pi}{4}$
y	0				1

b Use the trapezium rule with all the values of y in the completed table to obtain an estimate for the area of the shaded region R, giving your answer to 4 decimal places.

The region R is rotated through 2π radians around the x-axis to generate a solid of revolution.

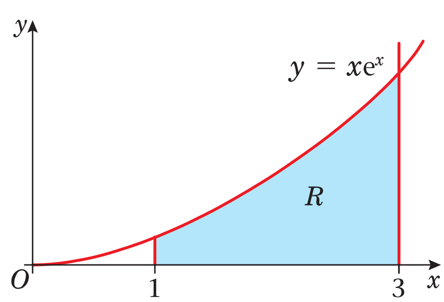
c Use integration to find an exact value for the volume of the solid generated. *E*

55 Using the substitution $u^2 = 2x - 1$, or otherwise, find the exact value of

$$\int_1^5 \frac{3x}{\sqrt{(2x - 1)}} \, dx.$$

E

56

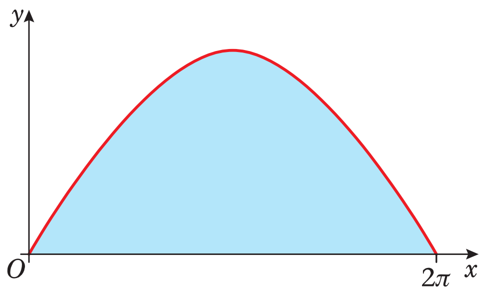


The figure shows the finite region R, which is bounded by the curve $y = xe^x$, the line $x = 1$, the line $x = 3$ and the x-axis.

The region R is rotated through 360 degrees about the x-axis.

Use integration by parts to find an exact value for the volume of the solid generated. *E*

57



The curve with equation $y = 3\sin\frac{x}{2}$, $0 \leqslant x \leqslant 2\pi$, is shown in the figure. The finite region enclosed by the curve and the x-axis is shaded.

a Find, by integration, the area of the shaded region.

This region is rotated through 2π radians about the x-axis.

b Find the volume of the solid generated. *E*

58 Use integration by parts to find the exact value of $\int_1^3 x^2 \ln x \, dx$. *E*

59 Use the substitution $u = 1 - x^2$ to find the exact value of

$$\int_0^{\frac{1}{2}} \frac{x^3}{(1-x^2)^{\frac{1}{2}}} \, dx.$$

60 **a** Express $\frac{5x+3}{(2x-3)(x+2)}$ in partial fractions.

b Hence find the exact value of

$$\int_2^6 \frac{5x+3}{(2x-3)(x+2)} \, dx,$$ giving your answer as a single logarithm. *E*

61 **a** Use integration by parts to find

$$\int x \cos 2x \, dx.$$

b Prove that the answer to part **a** may be expressed as

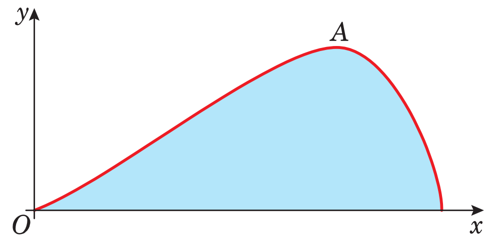
$$\tfrac{1}{2}\sin x\,(2x\cos x - \sin x) + C,$$

where C is an arbitrary constant. *E*

62 Use the substitution $u = 2^x$ to find the exact value of

$$\int_0^1 \frac{2^x}{(2^x+1)} \, dx.$$

63



The figure shows a graph of $y = x\sqrt{\sin x}$, $0 < x < \pi$.

The finite region enclosed by the curve and the x-axis is shaded as shown in the figure. A solid body S is generated by rotating this region through 2π radians about the x-axis. Find the exact value of the volume of S. *E*

64 **a** Find $\int x \cos 2x \, dx$.

b Hence, using the identity $\cos 2x = 2\cos^2 x - 1$, deduce $\int x \cos^2 x \, dx$. *E*

65 $\frac{2(4x^2+1)}{(2x+1)(2x-1)} = A + \frac{B}{(2x+1)} + \frac{C}{(2x-1)}$.

a Find the values of the constants A, B and C.

b Hence show that the exact value of

$$\int_1^2 \frac{2(4x^2+1)}{(2x+1)(2x-1)} \, dx \quad \text{is} \quad 2 + \ln k,$$

giving the value of the constant k. *E*

66 $f(x) = (x^2+1)\ln x$.

Find the exact value of $\int_1^e f(x) \, dx$. *E*

67 The curve C is described by the parametric equations

$$x = 3\cos t,\ y = \cos 2t, \qquad 0 \leqslant t \leqslant \pi.$$

Find a Cartesian equation of the curve C. *E*

68 The point $P(a, 4)$ lies on a curve C. C has parametric equations $x = 3t \sin t$, $y = 2 \sec t$, $0 \leqslant t < \frac{\pi}{2}$. Find the exact value of a. *E*

69 A curve has parametric equations

$$x = 2 \cot t,\ y = 2 \sin^2 t, \qquad 0 < t \leqslant \frac{\pi}{2}.$$

a Find an expression for $\frac{dy}{dx}$ in terms of the parameter t.

b Find an equation of the tangent to the curve at the point where $t = \frac{\pi}{4}$.

c Find a Cartesian equation of the curve in the form $y = f(x)$. State the domain on which the curve is defined. *E*

70 A curve has parametric equations

$$x = 7 \cos t - \cos 7t,$$

$$y = 7 \sin t - \sin 7t, \quad \frac{\pi}{8} < t < \frac{\pi}{3}.$$

a Find an expression for $\frac{dy}{dx}$ in terms of t. You need not simplify your answer.

b Find an equation of the normal to the curve at the point where $t = \frac{\pi}{6}$. Give your answer in its simplest exact form. *E*

71 A curve has parametric equations

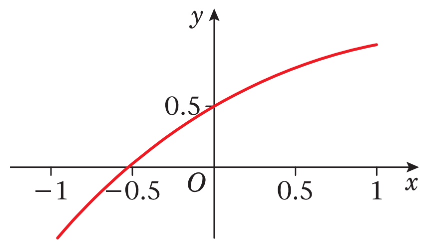
$$x = \tan^2 t,\ y = \sin t, \ 0 < t < \frac{\pi}{2}.$$

a Find an expression for $\frac{dy}{dx}$ in terms of t. You need not simplify your answer.

b Find an equation of the tangent to the curve at the point where $t = \frac{\pi}{4}$. Give your answer in the form $y = ax + b$, where a and b are constants to be determined.

c Find a Cartesian equation of the curve in the form $y^2 = f(x)$. *E*

72



The curve shown in the figure has parametric equations

$$x = \sin t,\ y = \sin\left(t + \frac{\pi}{6}\right), \ -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

a Find an equation of the tangent to the curve at the point where $t = \frac{\pi}{6}$.

b Show that a Cartesian equation of the curve is

$$y = \frac{\sqrt{3}}{2}x + \frac{1}{2}\sqrt{(1 - x^2)}, \ -1 < x < 1.$$

E

73 The curve C has parametric equations

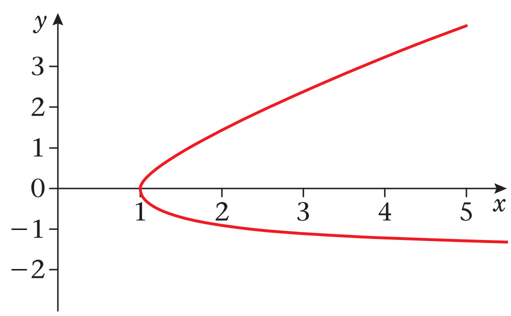
$$x = \frac{1}{1+t}, \quad y = \frac{1}{1-t}, \quad -1 < t < 1.$$

The line l is a tangent to C at the point where $t = \frac{1}{2}$.

a Find an equation for the line l.

b Show that a Cartesian equation for the curve C is $y = \frac{x}{2x - 1}$. *E*

74



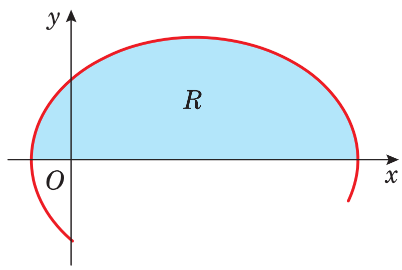
The curve shown has parametric equations $x = t + \frac{1}{t}$, $y = t - 1$ for $t > 0$.

a Find the value of the parameter t at each of the points where $x = 2\frac{1}{2}$.

b Find the gradient of the curve at each of these points.

c Find the area of the finite region enclosed between the curve and the line $x = 2\frac{1}{2}$. *E*

75



The curve shown in the figure has parametric equations

$$x = t - 2\sin t,$$
$$y = 1 - 2\cos t,\ 0 \leqslant t \leqslant 2\pi.$$

a Show that the curve crosses the x-axis where $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$.

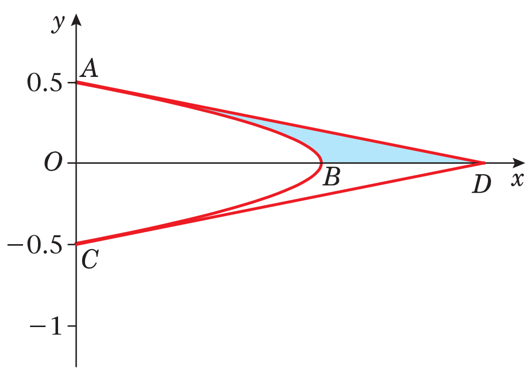
The finite region R is enclosed by the curve and the x-axis, as shown shaded in the figure

b Show that the area R is given by the integral

$$\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos t)^2\, dt.$$

c Use this integral to find the exact value of the shaded area. **E**

76



The curve shown in the figure has parametric equations

$$x = a\cos 3t,\ y = a\sin t,\ -\frac{\pi}{6} \leqslant t \leqslant \frac{\pi}{6}.$$

The curve meets the axes at points A, B and C, as shown.

The straight lines shown are tangents to the curve at the points A and C and meet the x-axis at point D. Find, in terms of a

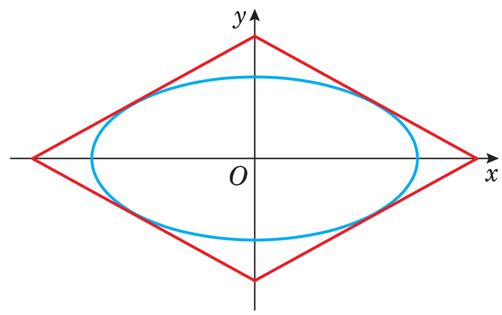
a the equation of the tangent at A,

b the area of the finite region between the curve, the tangent at A and the x-axis, shown shaded in the figure.

Given that the total area of the finite region between the two tangents and the curve is $10\,\text{cm}^2$

c find the value of a. **E**

77



A table top, in the shape of a parallelogram, is made from two types of wood. The design is shown in the figure. The area inside the ellipse is made from one type of wood, and the surrounding area is made from a second type of wood.

The ellipse has parametric equations,

$$x = 5\cos\theta,\ y = 4\sin\theta,\ \ 0 \leqslant \theta \leqslant 2\pi.$$

The parallelogram consists of four line segments, which are tangents to the ellipse at the points where

$\theta = \alpha,\ \theta = -\alpha,\ \theta = \pi - \alpha,\ \theta = -\pi + \alpha.$

a Find an equation of the tangent to the ellipse at $(5\cos\alpha, 4\sin\alpha)$, and show that it can be written in the form

$$5y\sin\alpha + 4x\cos\alpha = 20.$$

b Find by integration the area enclosed by the ellipse.

c Hence show that the area enclosed between the ellipse and the parallelogram is

$$\frac{80}{\sin 2\alpha} - 20\pi.$$ **E**

Practice paper

1 Use the binomial theorem to expand $\frac{1}{(2+x)^2}$, $|x| < 2$, in ascending powers of x, as far as the term in x^3, giving each coefficient as a simplified fraction. (6)

2 The curve C has equation

$$x^2 + 2y^2 - 4x - 6yx + 3 = 0$$

Find the gradient of C at the point (1, 3). (7)

3 Use the substitution $u = 5x + 3$, to find an exact value for

$$\int_0^3 \frac{10x}{(5x+3)^3}\,dx$$ (9)

4 **a** Find the values of A and B for which

$$\frac{1}{(2x+1)(x-2)} \equiv \frac{A}{2x+1} + \frac{B}{x-2}$$ (3)

b Hence find $\int \frac{1}{(2x+1)(x-2)}\,dx$, giving your answer in the form $\ln f(x)$. (4)

c Hence, or otherwise, obtain the solution of

$$(2x+1)(x-2)\frac{dy}{dx} = 10y,\ y > 0,\ x > 2$$

for which $y = 1$ at $x = 3$, giving your answer in the form $y = f(x)$. (5)

5 A population grows in such a way that the rate of change of the population P at time t in days is proportional to P.

a Write down a differential equation relating P and t. (2)

b Show, by solving this equation or by differentiation, that the general solution of this equation may be written as $P = Ak^t$, where A and k are positive constants. (5)

Initially the population is 8 million and 7 days later it has grown to 8.5 million.

c Find the size of the population after a further 28 days. (5)

6 Referred to an origin O the points A and B have position vectors $\mathbf{i} - 5\mathbf{j} - 7\mathbf{k}$ and $10\mathbf{i} + 10\mathbf{j} + 5\mathbf{k}$ respectively. P is a point on the line AB.

a Find a vector equation for the line passing through A and B. (3)

b Find the position vector of point P such that OP is perpendicular to AB. (5)

c Find the area of triangle OAB. (4)

d Find the ratio in which P divides the line AB. (2)